Normalized solutions to semilinear elliptic equations and systems

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Solutions with prescribed mass

Let $\Omega \subset \mathbb{R}^N$ be a bounded, Lipschitz domain, $1 < p < 2^* - 1$, $\rho > 0$.

Find $(U, \lambda) \in H_0^1(\Omega) \times \mathbb{R}$ s.t. $\begin{cases} -\Delta U + \lambda U = |U|^{p-1} U \\ \int_\Omega U^2 \, dx = \rho. \end{cases}$

Any u solution (for some λ) is a normalized solution.

Main goals:

- existence/non-existence, depending on p and ρ (and Ω);
- stability results for ground states (to be defined later).

Solutions with prescribed mass

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Warning: two critical exponents,

$$p = 1 + \frac{4}{N} \qquad (L^2\text{, or mass crit. exp.})$$

$$p = 2^* - 1 = 1 + \frac{4}{N-2} \qquad (\text{Sobolev, or energy crit. exp.})$$

For most of this talk: critical $:=$ L^2 critical

Motivation

Standing wave solutions of the (focusing) nonlinear Schrödinger equation (NLS)

$$-\Delta U + \lambda U = |U|^{p-1} U, \;\; U \in H^1_0(\Omega)$$

$$\Big\updownarrow \; \Phi(t,x) = e^{\mathrm{i}\lambda t} U(x)$$

$$\begin{cases} \mathrm{i}\frac{\partial \Phi}{\partial t} + \Delta \Phi + |\Phi|^{p-1}\Phi = 0 & (t,x) \in \mathbb{R} \times \Omega \\ \Phi(t,x) = 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases}$$

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NLS on bounded domains appears in different physical contexts:

- Nonlinear optics ($N = 2$, $p = 3$, $\Omega = \mathrm{disk}$): propagation of laser beams in hollow–core fibers. [Fibich, Merle (2001)]
- Bose–Einstein condensation ($N \leq 3$, $p = 3$): it models the presence of an infinite well trapping potential (to describe confined particles in quantum mechanics systems). [Lieb et al (2006), Bartsch, Parnet (2012)]

$p = 3$ **is subcritical for** $N = 1$**, critical for** $N = 2$ **and supercritical for** $N = 3$ (and Sobolev critical for $N = 4$).

Basic facts about NLS

$$\begin{cases} \mathrm{i}\dfrac{\partial \Phi}{\partial t} + \Delta\Phi + |\Phi|^{p-1}\Phi = 0 & (t,x) \in \mathbb{R} \times \Omega \\ \Phi(t,x) = 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases}$$

Conserved quantities along trajectories (at least formally):

- Energy: $\mathcal{E}(\Phi) = \displaystyle\int_\Omega \left(\frac{1}{2}|\nabla\Phi|^2 - \frac{1}{p+1}|\Phi|^{p+1} \right) dx$
- Mass (or Charge): $\mathcal{Q}(\Phi) = \displaystyle\int_\Omega |\Phi|^2 \, dx.$

Two points of view about standing waves...

Going back to the elliptic problem:

$$-\Delta U + \lambda U = |U|^{p-1}U, \qquad U \in H^1_0(\Omega).$$

Two points of view:

1. The chemical potential $\lambda \in \mathbb{R}$ is given
   - ▷ Solutions are critical points of the *Action Functional*:

$$\begin{aligned}\mathcal{A}_\lambda(U) &= \mathcal{E}(U) + \frac{\lambda}{2}\mathcal{Q}(U) \\ &= \frac{1}{2}\int_\Omega (|\nabla U|^2 + \lambda U^2)\,dx - \frac{1}{p+1}\int_\Omega |U|^{p+1}\,dx.\end{aligned}$$

Two points of view about standing waves...

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2. $\lambda \in \mathbb{R}$ is an unknown of the problem
   - ▷ Fix the mass $\mathcal{Q}(U) = \rho$, and find critical points of $\mathcal{E}|_{\{\mathcal{Q}=\rho\}}$.
   - ▷ λ appears as a Lagrange multiplier

Here we focus on the second point of view.

... and two notions of ground states

Consequently, ground states are defined by two (non equivalent) minimizations:

$$\text{either} \qquad \inf \mathcal{A}_\lambda \qquad\qquad \text{or} \qquad \inf\{\mathcal{E} : \mathcal{Q} = \rho\}.$$

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either $\inf \mathcal{A}_\lambda$ or $\inf\{\mathcal{E} : \mathcal{Q} = \rho\}$.

1. λ given. Least Action Solutions:
   - ▷ minimize the action $\mathcal{A}_\lambda$ among its nontrivial critical points.

$$a_\lambda = \inf\{\mathcal{A}_\lambda(U) : U \in H^1_0,\ U \not\equiv 0,\ \mathcal{A}'_\lambda(U) = 0\}$$

   [Berestycki, Lions ARMA (1983)]

2. λ unknown, fixed mass. Least Energy Solutions: $p \leq 1 + \frac{4}{N}$,

$$e_\rho = \inf_{\mathcal{Q}(U)=\rho} \mathcal{E}(U)$$

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Why the restriction on the exponent in the second case?

The L^2-critical exponent

This critical exponent comes from the Gagliardo-Nirenberg inequality:

$$\int_\Omega |U|^{p+1}\,dx \le C_{N,p}\left(\int_\Omega |\nabla U|^2\,dx\right)^{\beta_{N,p}}\left(\int_\Omega |U|^2\,dx\right)^{p+1-\beta_{N,p}} \qquad \forall U \in H^1_0(\Omega).$$

with $\beta_{N,p}$, $C_{N,p}$ independent of Ω. If $\mathcal{Q}(U)=\rho$:

$$\mathcal{E}(U) \ge \frac{1}{2}\int_\Omega |\nabla U|^2\,dx - C\left(\int_\Omega |\nabla U|^2\,dx\right)^{\beta}$$

where C depends on ρ and

$$\beta = \frac{N}{4}(p-1) \le 1 \qquad \iff \qquad p \le 1 + \frac{4}{N}.$$

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where C depends on ρ and

$$\beta = \frac{N}{4}(p-1) \le 1 \qquad \Longleftrightarrow \qquad p \le 1 + \frac{4}{N}.$$

Then

$$a_\lambda = \inf\{\mathcal{A}_\lambda(U) : U \text{ non-trivial critical point}\}$$
$$e_\rho = \inf\left\{\mathcal{E}(U) : \mathcal{Q}(U) = \rho, U \text{ constrained critical point}\right\}.$$

Straightforward remarks in $\mathbb{R}^N$

Let $\Omega = \mathbb{R}^N$ and $Z_{N,p}$ be the unique solution (up to translations) of

$$-\Delta Z + Z = Z^p, \quad Z \in H^1(\mathbb{R}^N), \quad Z > 0.$$

Then it “uniquely” achieves the Gagliardo-Nirenberg inequality.

Scaling:

$$U_h(x) = h^2 Z_{N,p}(h^{p-1}x) \quad \text{satisfies} \quad \begin{cases} -\Delta U_h + h^{2(p-1)} U_h = U^p, \\ \int_{\mathbb{R}^N} U_h^2 \, dx = h^{4-N(p-1)} \int_{\mathbb{R}^N} Z_{N,p}^2 \, dx. \end{cases}$$

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Then

1. a_λ is uniquely achieved for every $\lambda > 0$ (with $h = \lambda^{1/2(p-1)}$);
2. if $p \neq 1 + 4/N$, e_ρ is uniquely achieved for every $\rho > 0$;
3. in the critical case $p = 1 + 4/N$: e_ρ is achieved iff $\rho = \|Z_{N,p}\|_{L^2}^2$, by infinitely many solutions.

The L^2-critical exponent: consequences when $\Omega = \mathbb{R}^N$

$$\mathrm{i}\frac{\partial \Phi}{\partial t} + \Delta\Phi + |\Phi|^{p-1}\Phi = 0 \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^N$$

Critical Exponent: $\quad p = 1 + \dfrac{4}{N}.$

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- Subcritical Case ($1 < p < 1 + 4/N$): global existence for all initial data.
- Critical Case ($p = 1 + 4/N$): global existence for data with small mass $\mathcal{Q}$.
- Supercritical Case ($p > 1 + 4/N$): explosion in finite time.

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- Critical Case ($p = 1 + 4/N$): global existence for data with small mass $\mathcal{Q}$.
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Let $Z_{N,p}$ be the unique solution (up to translations) of

$$-\Delta Z + Z = Z^p, \quad Z \in H^1(\mathbb{R}^N), \quad Z > 0.$$

Then it achieves the Gagliardo-Nirenberg inequality and

- $e^{\mathrm{i}t} Z_{N,p}$ is orbitally stable if $p < 1 + 4/N$ (subcritical);
- $e^{\mathrm{i}t} Z_{N,p}$ is unstable if $p \geq 1 + 4/N$ (critical and supercritical).

Proofs: [Coffman (1972), Kwong ARMA (1989)], [Cazenave, Lions CMP (1982)].

Known results

- When global minimizers do not exist and scaling is not allowed, existence of normalized solutions is nontrivial
- many techniques developed for the case with fixed λ can not be directly adapted

After

[Jeanjean, *Nonlinear Anal* (1997)]

only more recent results:

- normalized solutions in $\mathbb{R}^N$, with non-homogeneous nonlinearities and/or systems: Bartsch, Bellazzini, de Valeriola, Guo, Jeanjean, Soave
- on bounded domains (both equations and systems): Noris, Pistoia, Pellacci, Pierotti, Tavares, Vaira, V.
- on metric graphs: Adami, Dovetta, Serra, Tilli
- for Mean Field Games systems: Cesaroni, Cirant, Gomes, V.

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Warning: we restrict our attention to positive solutions!

Theorem ([Noris, Tavares, V., Analysis & PDE (2014)])

Let $\Omega = B_1$.

1. *If* $1 < p < 1 + 4/N$ *then, for every* $\rho > 0$, *there exists a unique positive solution, which achieves* e_ρ;

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Theorem ([Noris, Tavares, V., Analysis & PDE (2014)])

Let $\Omega = B_1$.

1. *If* $1 < p < 1 + 4/N$ *then, for every* $\rho > 0$*, there exists a unique positive solution, which achieves* e_ρ;
2. *if* $p = 1 + 4/N$,
   - *for* $0 < \rho < \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$*, there exists a unique positive solution, which achieves* e_ρ;
   - *for* $\rho \geq \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$*, no positive solution exists;*

Existence

Case $\Omega = B_1$

Warning: we restrict our attention to positive solutions!

Theorem ([Noris, Tavares, V., Analysis & PDE (2014)])

Let $\Omega = B_1$.

1. *If $1 < p < 1 + 4/N$ then, for every $\rho > 0$, there exists a unique positive solution, which achieves e_ρ;*
2. *if $p = 1 + 4/N$,*
   - *for $0 < \rho < \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$, there exists a unique positive solution, which achieves e_ρ;*
   - *for $\rho \geq \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$, no positive solution exists;*
3. *if $1 + 4/N < p < 2^* - 1$, there exists $\rho^* > 0$, depending on p, such that:*
   - *e_ρ is achieved if and only if $0 < \rho \leq \rho^*$.*
   - *no positive solutions exists for $\rho > \rho^*$,*
   - *For $0 < \rho < \rho^*$ there exist at least two distinct positive solutions.*

   In this latter case, there exist positive solutions of the problem which are not least energy solutions.

Remarks on least action solutions

$$-\Delta U + \lambda U = U^p, \qquad U \in H^1_0(B_1)$$

$$a_\lambda = \inf\{\mathcal{A}_\lambda(U) : U \in H^1_0(B_1),\ U \not\equiv 0,\ \mathcal{A}'_\lambda(U) = 0\}$$

Existence and uniqueness of (positive) solution for any $\lambda \in (-\lambda_1(B_1), \infty)$:

$$\lambda \mapsto U_\lambda \qquad \text{(least action solution).}$$

Corollary

- *For $1 < p \leq 1 + 4/N$ (subcritical and critical), the notions of least energy and of least action coincide.*
- *For $p > 1 + 4/N$, this no longer happens: there are least action solutions which are not least energy solutions.*

Orbital Stability

Let U denote a **least energy solution** of

$$\begin{cases} -\Delta U + \lambda U = U^p \\ \int_{B_1} U^2 \, dx = \rho, \quad U > 0. \end{cases}$$

and let

$$\Phi(t,x) = e^{\mathrm{i}\lambda t} U(x).$$

Theorem

- *If* $1 < p \le 1 + 4/N$ *(subcritical and critical) then* Φ *is orbitally stable;*
- *if* $1 + 4/N < p < 2^* - 1$ *(supercritical) then* Φ *is orbitally stable for a.e.* $\rho \in (0, \rho^*]$.

Conjecture: in the latter case, stability $\forall\ \rho \in (0, \rho^*)$, instability for $\rho = \rho^*$.

The boundary of the domain has a stabilizing effect.

This effect was already observed in
[Fibich, Merle Phys. D (2001), Fukuizumi, Selem, Kikuchi Nonlinearity (2012)]
when dealing with least action solutions:
according to their results, the corresponding standing waves are stable when $\lambda \sim -\lambda_1(B_1)$ and $\lambda \sim +\infty$, in the subcritical and critical cases.
In the supercritical one

- when $\lambda \sim -\lambda_1$, least action solutions are stable;
- when $\lambda \sim +\infty$, least action solutions are unstable.

Our contribution:

Corollary

Let U_λ be the unique positive solution of

$$-\Delta U + \lambda U = U^p, \qquad U \in H_0^1(B_1).$$

(hence a least action solution).
If $1 < p \leq 1 + 4/N$, then $e^{i\lambda t} U_\lambda$ is orbitally stable for every $\lambda \in (-\lambda_1(B_1), +\infty)$.

Ideas of the proof

For the moment, take Ω any bounded domain.

$$\begin{cases} -\Delta U + \lambda U = U^p \\ \displaystyle\int_\Omega U^2\,dx = \rho \end{cases} \qquad \xleftrightarrow[\mu=\rho^{(p-1)/2}]{U(x)=\sqrt{\rho}u(x)} \qquad \begin{cases} -\Delta u + \lambda u = \mu u^p \\ \displaystyle\int_\Omega u^2\,dx = 1 \end{cases}$$

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We choose to parameterize solutions with

$$\alpha = \int_\Omega |\nabla u|^2 \geq \lambda_1(\Omega),$$

and to study the (possibly ill-defined, multivalued) map

$$\boxed{\alpha \mapsto (u, \lambda, \mu).}$$

If we succeed, our original problem is translated into:

to find α such that a corresponding $\mu = \rho^{(p-1)/2}$.

Existence

Optimization Problem with two constraints

For each $\alpha > \lambda_1(\Omega)$, take

$$M_\alpha = \sup\left\{ \int_\Omega |u|^{p+1}\,dx : \ u \in H_0^1(\Omega), \int_\Omega u^2\,dx = 1, \ \int_\Omega |\nabla u|^2\,dx = \alpha \right\}.$$

For each $\alpha > \lambda_1(\Omega)$, take

$$M_\alpha = \sup\left\{ \int_\Omega |u|^{p+1}\,dx : \ u \in \ H^1_0(\Omega), \int_\Omega u^2\,dx = 1, \ \int_\Omega |\nabla u|^2\,dx = \alpha \right\}.$$

Related to Gagliardo-Nirenberg inequality:

$$\|u\|^{p+1}_{L^{p+1}(\Omega)} \le C_{N,p} \|u\|^{p+1-N(p-1)/2}_{L^2(\Omega)} \|\nabla u\|^{N(p-1)/2}_{L^2(\Omega)}, \qquad \forall u \in H^1_0(\Omega)$$

by:

$$C_{N,p} = \sup_{\alpha \ge \lambda_1(\Omega)} \frac{M_\alpha}{\alpha^{N(p-1)/2}}.$$

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by:

$$C_{N,p} = \sup_{\alpha \geq \lambda_1(\Omega)} \frac{M_\alpha}{\alpha^{N(p-1)/2}}.$$

Theorem

M_α is achieved by a positive function $u \in H_0^1(\Omega)$, and there exist $\mu > 0$, $\lambda > -\lambda_1(\Omega)$ such that

$$-\Delta u + \lambda u = \mu u^p, \quad \int_\Omega u^2\,dx = 1, \quad \int_\Omega |\nabla u|^2\,dx = \alpha.$$

Behavior near $\alpha = \lambda_1$

What happens for $\alpha \sim \lambda_1(\Omega)$?

$$-\Delta u + \lambda u = \mu u^p, \qquad \int_\Omega u^2\,dx = 1.$$

Take the map:

$$\Phi(u,\mu,\lambda) = \left(\Delta u - \lambda u + \mu u^p,\ \int_\Omega u^2\,dx - 1,\ \int_\Omega |\nabla u|^2\,dx\right).$$

We can prove that $(\varphi_1, 0, -\lambda_1)$ is ordinary singular for Φ, in the sense of Ambrosetti–Prodi.

Proposition

The equation

$$\Phi(u,\mu,\lambda) = (0,0,\lambda_1+\varepsilon)$$

has exactly two solutions for each small $\varepsilon > 0$. One of these solutions is such that $\int_\Omega u^{p+1} = M_\alpha$ and satisfies $\lambda > -\lambda_1$, $\mu > 0$. The other is associated to the defocusing case $\lambda < -\lambda_1$, $\mu < 0$.

Behavior for α large

By construction, u has Morse index 1 or 2. When $\alpha_n \to +\infty$, we find a singularly perturbed problem: if x_n is a local maximum for u_n, then

$$v_n(x) := \left(\frac{\mu_n}{\lambda_n}\right)^{1/(p-1)} u_n\left(\frac{x}{\sqrt{\lambda_n}} + x_n\right)$$

satisfies, up to subs.,

$$v_n \to Z_{N,p} \qquad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^N).$$

[Druet, Hebey and Robert; Esposito and Petralla]. As a consequence

Proposition

As $\alpha_n \to +\infty$, we have

$$\lambda_n \to +\infty,$$

and

$$\mu_n \to \begin{cases} +\infty & 1 < p < 1 + \frac{4}{N} \\ C(N,p) & p = 1 + \frac{4}{N} \\ 0 & 1 + \frac{4}{N} < p < 2^* - 1. \end{cases}$$

The upper curve

$\Omega = B_1(0)$

Back to $\Omega = B_1(0)$ (and $\mu > 0$): combining results/ideas of many authors [Gidas Ni Nirenberg, Kwong, Kwong Li, Korman, Aftalion Pacella, Felmer Martínez Tanaka] we have uniqueness of positive solutions, which Morse index is always one. Let:

$$\mathcal{S}^+ = \left\{ (u, \mu, \lambda) : -\Delta u + \lambda u = \mu u^p,\ u > 0,\ \int_{B_1} u^2 = 1,\ \mu > 0 \right\}$$

Proposition

We can parameterize $\mathcal{S}^+$ with α in a smooth way:

$$\alpha \mapsto (u(\alpha), \mu(\alpha), \lambda(\alpha)),$$

where $\int_{B_1} |\nabla u(\alpha)|^2 = \alpha$, and $u(\alpha)$ achieves M_α. Moreover, $\lambda'(\alpha) > 0$.

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where $\int_{B_1} |\nabla u(\alpha)|^2 = \alpha$, and $u(\alpha)$ achieves M_α. Moreover, $\lambda'(\alpha) > 0$.

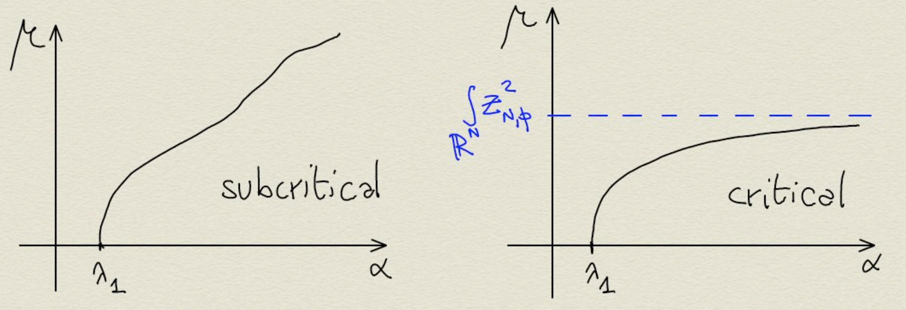
It turns out that the behavior of μ is crucial:

- prescribing the mass ρ is equivalent to prescribing $\mu = \rho^{(p-1)/2}$;
- μ' positive (resp. negative) implies orbital stability (resp. instability) of the corresponding standing waves [Grillakis, Shatah, Strauss JFA (1987)].

The upper curve

$\Omega = B_1(0)$

Subcritical and Critical cases. We prove $\mu' > 0$ for every α

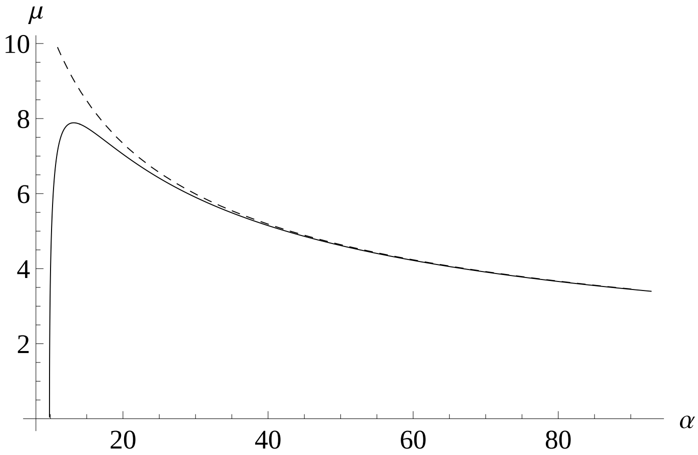


Existence and stability follow.

The upper curve

$\Omega = B_1(0)$

Supercritical case, $p > 1 + 4/N$: Conjecture



Numerical simulation for $N = 3$, $p = 3$

Existence follows. Also stability, with some further work.

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Some questions

The full picture for positive solutions in the ball suggests several questions/conjectures (for p critical or supercritical):

- non-existence of positive solutions for large ρ in general Ω
- existence of positive solutions, maybe stable, for small ρ in general Ω
- existence of non-necessary positive solutions for large ρ (also in B)
- systems of NLS equations on bounded domains.

Non-existence: some examples

For p critical and supercritical and ρ large no positive solution exists in the ball.

What about general domains and/or changing-sign solutions?

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What about general domains and/or changing-sign solutions?

- Necklace solutions: any Dirichlet solution of

$$-\Delta U + \lambda U = |U|^{p-1}U \qquad \text{in a rectangle } R \subset \mathbb{R}^N$$

can be scaled to a solution of

$$-\Delta U + k^2\lambda U = |U|^{p-1}U \qquad \text{in } R/k,\ k \in \mathbb{N}_+,$$

and then k^N copies of it can be juxtaposed, with alternating sign. The new solution on R has $k^{4/(p-1)}$ times the mass of the starting one. In the disk?

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What about general domains and/or changing-sign solutions?

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  $$-\Delta U + \lambda U = |U|^{p-1}U \qquad \text{in a rectangle } R \subset \mathbb{R}^N$$

  can be scaled to a solution of

  $$-\Delta U + k^2 \lambda U = |U|^{p-1}U \qquad \text{in } R/k,\ k \in \mathbb{N}_+,$$

  and then k^N copies of it can be juxtaposed, with alternating sign. The new solution on R has $k^{4/(p-1)}$ times the mass of the starting one. In the disk?

- Dumb-bell domains: taking k copies of B_1, joined by small channels, one can construct positive solutions having mass close to k times the mass of the solution on B_1.

Non-existence: some examples

For p critical and supercritical and ρ large no positive solution exists in the ball.

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The second example suggests to classify solutions in terms of their Morse index, rather than in terms of their nodal properties.

Non-existence: the role of the Morse index

U solution for some λ. Its Morse index is

$$m(U) = \max \left\{ k : \begin{array}{c} \exists V \subset H^1_0(\Omega),\ \dim(V) = k : \forall v \in V \setminus \{0\} \\ \displaystyle\int_\Omega |\nabla v|^2 + \lambda v^2 - p|U|^{p-1} v^2 \, dx < 0 \end{array} \right\} \in \mathbb{N}.$$

In $\Omega = B_1$, a solution U is positive iff $m(U) = 1$.

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Theorem ([Pierotti, V., Calc Var PDE (2017)])

For every $\Omega \subset \mathbb{R}^N$ bounded C^1 domain, $k \geq 1$, $1 < p < 2^* - 1$,

$$\sup \left\{ \rho > 0 : \begin{array}{l} \textit{there exists a solution } U \textit{ (for some } \lambda) \\ \textit{having Morse index } m(U) \leq k \end{array} \right\} < +\infty \iff p \geq 1 + \frac{4}{N}.$$

Proof: blow-up analysis of sequences of solutions with bounded Morse index, via suitable a priori pointwise estimates. In case the mass is unbounded on such a sequence, the sequence splits in the superposition of at most k profiles, which converge (suitably rescaled) to entire solutions.

Existence: Local minimizers

The Grillakis-Shatah-Strauss theory for orbital stability implies that the orbitally stable solutions we found in B are minimizers.

Theorem ([Pierotti, V., Calc Var PDE (2017)], [Noris, Tavares, V., Nonlinearity (2019)])

For every $0 < \rho < \hat{\rho}_1 = \hat{\rho}_1(\Omega, p)$ there exists a solution which is a local minimizer of the energy $\mathcal{E}$ on $\{\mathcal{Q} = \rho\}$, and the corresponding ground state set is orb. stable. Furthermore, for every Lipschitz Ω,

- $1 < p < 1 + \dfrac{4}{N} \implies \hat{\rho}_1(\Omega, p) = +\infty$,
- $p = 1 + \dfrac{4}{N} \implies \hat{\rho}_1(\Omega, p) \geq \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$,
- $1 + \dfrac{4}{N} < p \leq 2^* - 1 \implies \hat{\rho}_1(\Omega, p) \geq D_{N,p}\lambda_1(\Omega)^{\frac{2}{p-1} - \frac{N}{2}}$,

where the universal constant $D_{N,p}$ is explicit.

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- We deal also with the Sobolev critical case.
- This explains the second (unstable) positive solution in the supercritical case.
- The last estimate is new also for the ball. Furthermore, it provides information on the necklace solutions.

Consequences on special domains

Theorem

Let $\Omega = B$ be a ball in $\mathbb{R}^N$. Then

$$p < 1 + \frac{4}{N-1} \quad \Longrightarrow \quad \text{there exists a solution for every } \rho > 0.$$

An analogous result holds when $\Omega = R$ is a rectangle, without further restrictions on $p < 2^* - 1$.

Higher masses require higher Morse index–solutions. In particular, in the ball, even though no positive solution exists, nodal solutions with higher Morse index can be obtained: nodal ground states with higher Morse index.

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Proof: divide the ball in $2k$ equal sectors and estimate the corresponding first eigenvalue.

Systems

The existence of minimizers and their orbital stability can be proved also for systems:

$$\begin{cases} -\Delta u_1 + \omega_1 u_1 = \mu_1 u_1 |u_1|^{p-1} + \beta u_1 |u_1|^{(p-3)/2} |u_2|^{(p+1)/2} \\ -\Delta u_2 + \omega_2 u_2 = \mu_2 u_2 |u_2|^{p-1} + \beta u_2 |u_2|^{(p-3)/2} |u_1|^{(p+1)/2} \\ \int_\Omega u_i^2 = \rho_i, \quad i = 1, 2, \\ (u_1, u_2) \in H_0^1(\Omega; \mathbb{R}^2). \end{cases}$$

Also in this case we can cover $p = 2^* - 1$.
[Noris, Tavares, V., Nonlinearity (2019)]

Open problems – Work in progress

- Our main conjecture is the existence of solutions for every ρ, p, Ω.
- Methodological approach: topological approach? Indeed, applications to ergodic Mean Field Games systems motivate the study of normalized solutions to some class of non-variational semilinear elliptic equations/systems.
- Metric graphs: existence of normalized local minimizers when a global one does not exists.
- Semiclassical analysis: it seems to make sense only in the subcritical case $p < 1 + \frac{4}{N}$.

Thank you for your attention, and...

Tanti auguri Professore!!