Maximal solution of the Liouville equation in doubly connected domain

Giusi Vaira

Variational methods, with applications to problems in mathematical physics and geometry
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The problem

Let us consider the following Liouville type problem:

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\Delta u = \lambda^2 e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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- $\Omega \subset \mathbb{R}^2$ is a smooth and bounded domain.
- $\lambda > 0$ is a parameter.

Asymptotic behavior of solutions of $(\mathcal{P}_\lambda)$

Let u_λ be a solution of $(\mathcal{P}_\lambda)$.

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$$\lambda^2 \int_\Omega e^{u_\lambda}\, dx \to 8\pi\ell \qquad as\ \lambda \to 0 \quad \boxed{\ell = 0, 1, 2, \dots, +\infty}$$

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(b) $\ell \in \mathbb{N} \implies$ there exist ℓ different points $\xi_1, \ldots, \xi_\ell \in \Omega$ s.t. as $\lambda \to 0$ $u_\lambda(\xi_j) \to +\infty$ and

$$u_\lambda \to 8\pi \sum_{j=1}^{\ell} G(x, \xi_j) \qquad C^2_{loc}(\bar\Omega \setminus \{\xi_1, \ldots, \xi_\ell\})^1$$

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Existence results for $(\mathcal{P}_\lambda)$

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About the *minimal* solution

$u_{\lambda,min}$ is a solution close to zero which represents a strict local minimizer of the energy functional

$$\boxed{J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \lambda^2 \int_\Omega e^u\,dx}$$

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About the *unbounded* solution

- If **u_λ blows-up at ℓ points $\xi_1,\dots,\xi_\ell$** then $(\xi_1,\dots,\xi_\ell)$ is a critical point of the Kirchoff-Routh path function

$$\mathcal{H}(x_1,\dots,x_\ell) := \sum_{j=1}^{\ell} H(x_j,x_j) - \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} G(x_i,x_j)$$

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G is the Green's function of $-\Delta$ with D.B.C. and H is its regular part

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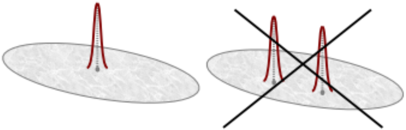
Nagasaki, Suzuki *Asymp. Anal. (1990)*

Baraket, Pacard *Calc. Var. (1997)*

del Pino, Kowalczyk, Musso *Calc. Var. (2005)*

Esposito, Grossi, Pistoia *Ann. I. H. P. (2005)*

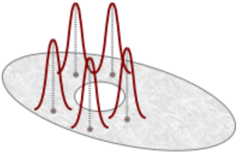
Situation in a convex domain



In a convex domain only solutions with one blow-up point do exist

Grossi, Takahashi *JFA (2010)*

Situation in a non-simply connected domain



In a non-simply connected domain for any $\ell \in \mathbb{N}$ there exists a solution with ℓ blow-up points

del Pino, Kowalczyk, Musso *Calc. Var. (2005)*

Existence results for $(\mathcal{P}_\lambda)$

Ω bounded domain	*minimal solution* $u_{\lambda,min}$ *unbounded solution* $u_{\lambda,unb}$
Ω bounded + convex	$u_{\lambda,unb}$ is a 1- *bubble solution*
Ω bounded + not simply connected	$u_{\lambda,unb}$ is a ℓ- *bubble solution*
$\Omega = \mathcal{A} = B_{r_2} \setminus B_{r_1}, r_2 > r_1$	*maximal solution* $u_{max,\lambda}$

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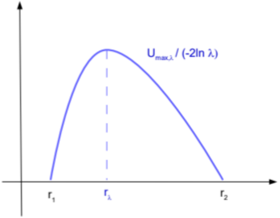
Nagasaki, Suzuki, *J. of Diff. Eqs.* (1990)

An accurate asymptotic analysis in the annulus

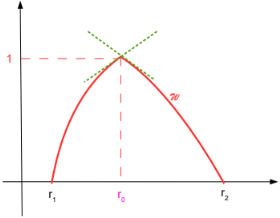
$\mathcal{A} := \{x \in \mathbb{R}^2 \, : \, r_1 < |x| < r_2\}$.
As $\lambda \to 0$

$$u_{max,\lambda} \sim \left(2 \ln \frac{1}{\lambda}\right) \mathcal{W} \text{ in } C^0_{loc}(r_1, r_2) \setminus r_0$$



where $r_0 := \sqrt{r_1 r_2}$ and $\mathcal{W}$ is the capacity potential of the curve $\gamma := \{|x| = r_0\}$, i.e. solves the elliptic problem

$$\begin{cases} \mathcal{W}'' + \frac{1}{r}\mathcal{W}' = 0 \; r \in (r_1, r_2) \setminus r_0 \\ \mathcal{W}(r_1) = \mathcal{W}(r_2) = 0 \\ \mathcal{W}(r_0) = 1 \end{cases}$$



and satisfies the free-boundary condition

$$\boxed{\mathcal{W}'_+(r_0) = -\mathcal{W}'_-(r_0)}$$

If r_λ is the maximum of the solution $u_{max,\lambda}(r_\lambda) := \|u_{max,\lambda}\|_\infty$ then

$$\bar{u}_\lambda(t) := u_{max,\lambda}(\varepsilon_\lambda t + r_\lambda) - u_{max,\lambda}(r_\lambda)$$

$$t \in \mathcal{A}_\lambda := \left(\frac{r_1 - r_\lambda}{\varepsilon_\lambda}, \frac{r_2 - r_\lambda}{\varepsilon_\lambda} \right)$$

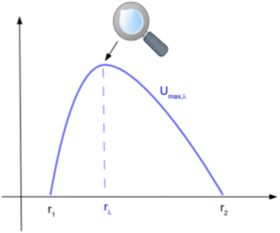
As $\lambda \to 0$

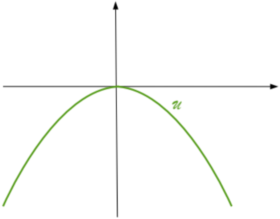
- $r_\lambda \to r_0 := \sqrt{r_1 r_2}$
- $\varepsilon_\lambda^2 := \frac{1}{\lambda^2 e^{\|u_{max,\lambda}\|_\infty}} \to 0$
- $\bar{u}_\lambda \to \mathcal{U}$ in $C^1_{loc}(\mathbb{R})$ where

$$\mathcal{U}(t) := \ln 4 \frac{e^{\sqrt{2}t}}{(1 + e^{\sqrt{2}t})^2}$$

is the unique solution to

$$\begin{cases} \mathcal{U}'' + e^{\mathcal{U}} = 0 \text{ in } \mathbb{R} \\ \mathcal{U}(0) = \mathcal{U}'(0) = 0 \end{cases}$$





Gladiali, Grossi, *Asymp. Anal.* (2007).

In a convex bounded domain there are only two solutions

- the minimal solution;
- the solution blowing up at the minimum point of the Robin's function $\mathcal{H}(x) := H(x, x)$.

Weston, *Siam J. Math. Anal. (1978)*

Moseley, *SIAM J. of Math. Anal. (1983)*

Suzuki, *Annales de l'I.H.P. (1992)*

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- the maximal solutions.

Open problem

Is it possible to find a *maximal solution* with infinite mass, i. e.

$$\int_\Omega \lambda^2 e^{u_\lambda} \to +\infty \qquad \text{as } \lambda \to 0$$

in a general domain?

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in a general domain?

Remark

A maximal solution cannot concentrate on a finite set of points

The main ingredient: the curve

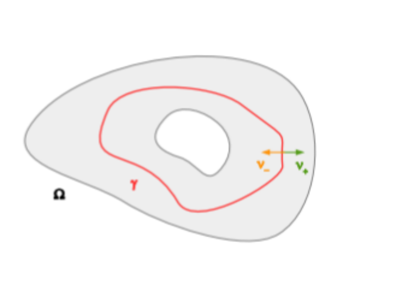
Let Ω be a bounded domain. Find $\gamma \subset \Omega$ a simple and closed curve whose capacity potential $W_\gamma \in C^2(\Omega \setminus \gamma) \cap C^0(\Omega)$

$$\begin{aligned} \Delta W_\gamma &= 0 && \text{in } \Omega \setminus \gamma \\ W_\gamma &= 0 && \text{on } \partial\Omega \\ W_\gamma &= 1 && \text{on } \gamma. \end{aligned}$$

satisfies the *free boundary condition*

$$\partial_{\nu^+} W_\gamma = \partial_{\nu^-} W_\gamma \qquad \text{on } \gamma$$

We call $(\mathcal{FB})$ this problem!



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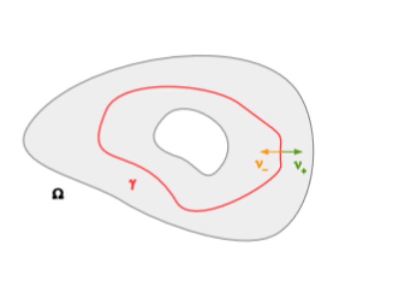

About $(\mathcal{F}\mathcal{B})$

In the Annulus

$\gamma := \{|x| = \sqrt{r_1 r_2} = r_0\}$ is the solution of the free boundary problem

$$\begin{cases} \mathcal{W}'' + \frac{1}{r}\mathcal{W}' = 0 \text{ if } r \in (r_1, r_2) \setminus r_0 \\ \mathcal{W}(r_1) = \mathcal{W}(r_2) = 0 \\ \mathcal{W}(r_0) = 1 \\ \mathcal{W}'_+(r_0) = -\mathcal{W}'_-(r_0) \end{cases}$$

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In a multiply connected domain

There exists γ solutions of $(\mathcal{FB})$?

About $(\mathcal{FB})$ in a doubly connected domain

Lemma

Let $\Omega \subset \mathbb{R}^2$ be a doubly connected set, bounded such that the bounded component of $\mathbb{R}^2 \setminus \Omega$ is not a point. There exists a simple, closed and smooth curve $\gamma \subset \Omega$ such that W_γ satisfies $(\mathcal{F}B)$. Additionally, γ is unique.

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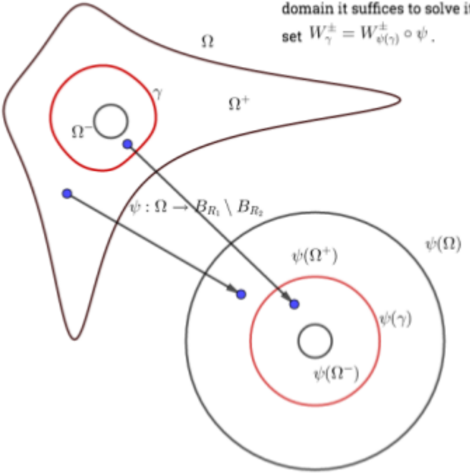
Let $\Omega \subset \mathbb{R}^2$ be a doubly connected set, bounded such that the bounded component of $\mathbb{R}^2 \setminus \Omega$ is not a point. There exists a simple, closed and smooth curve $\gamma \subset \Omega$ such that W_γ satisfies $(\mathcal{F}B)$. Additionally, γ is unique.

Remark

We rely here on a generalization of the Riemann mapping theorem which says that any doubly connected domain is conformally equivalent to an annulus, i.e. there exists a holomorphic, bijective map $\psi : \Omega \to B_{r_1} \setminus B_{r_2}$ with some $r_1 > r_2 > 0$.
Moreover we have that

$$\psi(\gamma) = \gamma_{r_0} \quad r_0 = \sqrt{r_1 r_2}.$$

To solve the problem in a general doubly connected domain it suffices to solve it in $\psi(\Omega)$ and set $W_\gamma^\pm = W_{\psi(\gamma)}^\pm \circ \psi$.



Theorem

Let Ω be a doubly connected, bounded domain such that the bounded component of $\mathbb{R}^2 \setminus \Omega$ is not a point. There exist a sequence $\lambda_n \to 0$, and a sequence of maximal solutions u_{λ_n} of the Liouville problem $(\mathcal{P}_\lambda)$ with the following properties:

1. It holds
$$\frac{u_{\lambda_n}}{2\log\frac{1}{\lambda_n}} \to W_\gamma, \qquad \text{as } \lambda_n \to 0$$
over compact subsets of $\Omega \setminus \gamma$.
2. We have
$$\frac{\lambda_n^2}{2\log\frac{1}{\lambda_n}} \int_\Omega e^{u_{\lambda_n}}\,dx \to \frac{4\pi}{\log\frac{r_1}{r_2}}, \quad \text{as } \lambda_n \to 0.$$

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Some remarks

- The mass of the maximal solution depends only on the conformal class of Ω.
- In the general case we can only prove existence of the maximal solution for an open set of λ such that 0 is its limit point, while in the radially symmetric case the analogous theorem holds for the interval $\lambda \in (0, \lambda_0)$ for some λ_0.
- The Morse index of the maximal solution grows to $+\infty$ as $\lambda \to 0$.

How to build the solution

Glue the two profiles founded by Gladiali and Grossi

Scheme for the annulus

Far from the curve $\gamma := \{|x| = \sqrt{r_1 r_2}\}$

$$u_\lambda \sim 2\ln\frac{1}{\lambda}\mathcal{W}$$

where

$$\begin{cases} \mathcal{W}'' + \frac{1}{r}\mathcal{W}' = 0 \text{ if } r \in (r_1, r_2) \setminus r_0 \\ \mathcal{W}(r_1) = \mathcal{W}(r_2) = 0 \\ \mathcal{W}(r_0) = 1 \\ \mathcal{W}'_+(r_0) = -\mathcal{W}'_-(r_0) \end{cases}$$

Close to the curve $\gamma := \{|x| = \sqrt{r_1 r_2}\}$

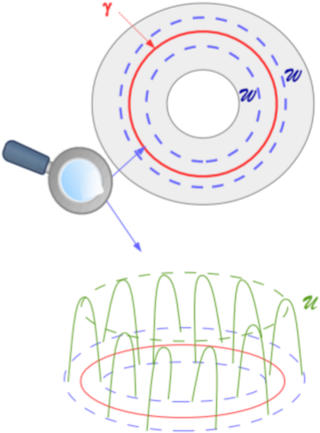
the solutions looks like

$$u_\lambda \to \mathcal{U}$$

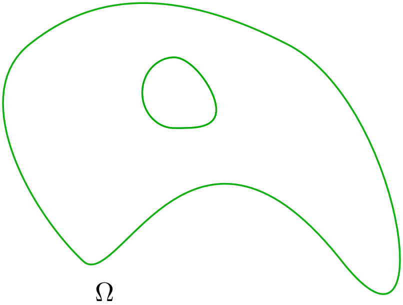
$$\mathcal{U}(t) := \ln 4\frac{e^{\sqrt{2}t}}{(1+e^{\sqrt{2}t})^2}$$

is the unique solution to

$$\begin{cases} \mathcal{U}'' + e^{\mathcal{U}} = 0 \text{ in } \mathbb{R} \\ \mathcal{U}(0) = \mathcal{U}'(0) = 0. \end{cases}$$



Outline of the proof



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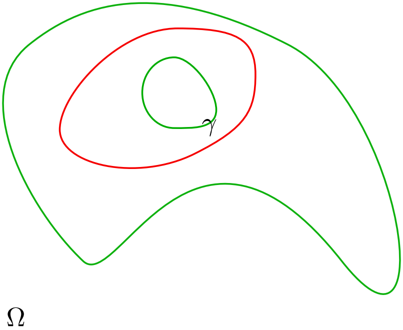

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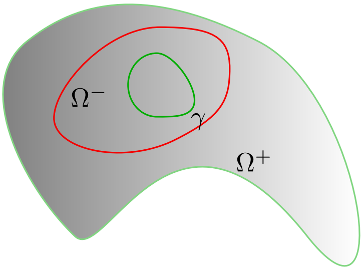
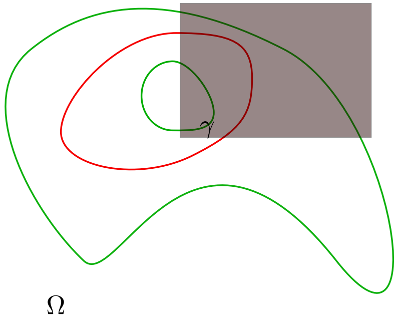


$$\Omega = \Omega^+ \cup \Omega^-$$

Outline of the proof



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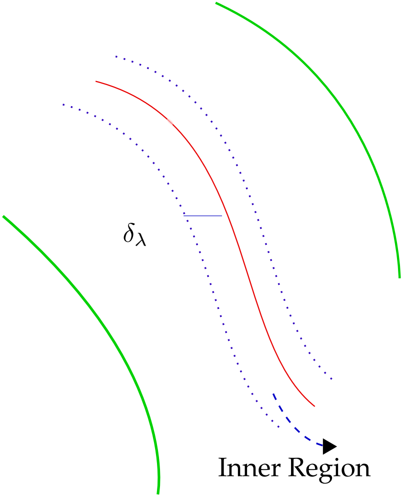


We parametrize a neighborhood of γ by $x \mapsto (s, t)$ where s is the arc length parameter on and t is the signed distance. Near γ the solution should be of the form

$$v_0(s,t) = \mathcal{U}(\lambda\mu_\lambda(s)t) + 2\log\mu_\lambda(s)$$

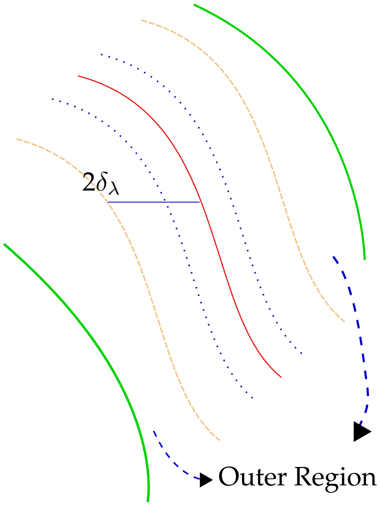
where $\frac{1}{\lambda\mu_\lambda} \to 0$ as $\lambda \to 0$ and

$$\mathcal{U}(t) = \ln 4\frac{e^{\sqrt{2}t}}{(1+e^{\sqrt{2}t})^2}$$

and

$$\delta_\lambda \to 0 \quad \text{as } \lambda \to 0$$

Outline of the proof



Far from γ the function u is essentially harmonic and the outer approximation of u is given by

$$(*) \begin{cases} \Delta w_0^\pm = 0 & \text{in } \Omega^\pm \\ w_0^\pm = 0 & \text{in } \partial\Omega \cap \partial\Omega^\pm \end{cases}$$

No information on γ so far.

Outline of the proof

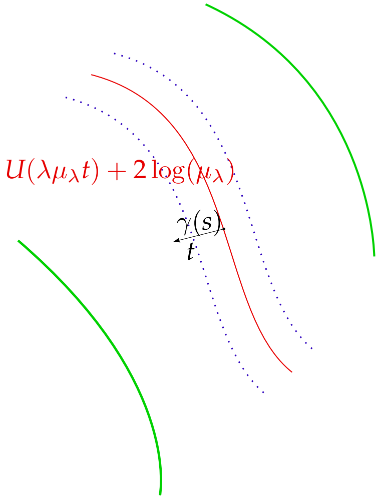


Figure: Inner approximation

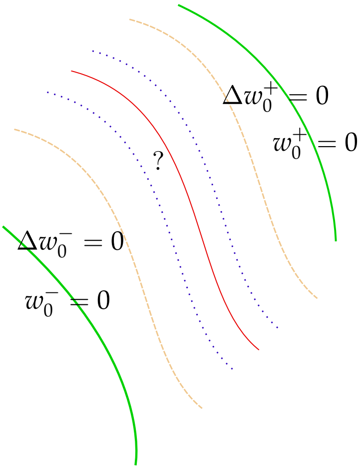
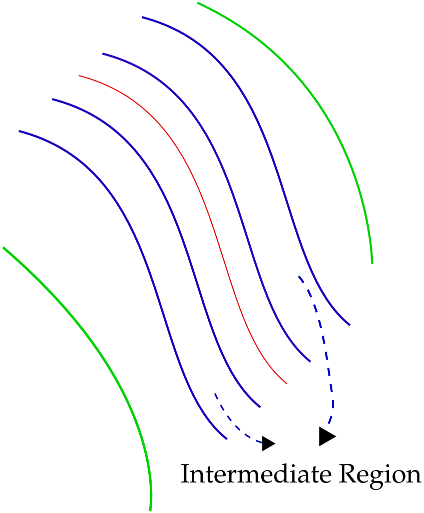


Figure: Outer approximation

Outline of the proof



The matching conditions

It holds:

$$\boxed{v_0(s,t) = b_0 + 2\log\mu_\lambda - a_0\lambda\mu_\lambda|t| + O\left(e^{-a_0\lambda\mu_\lambda|t|}\right)}$$

and

$$\boxed{w_0^\pm(s,t) = w_0^\pm(s,0) + t\partial_n w_0^\pm(s,0) + \dots}$$

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$$(**)\begin{cases} w_0^+ = b_0 + 2\log\mu_\lambda \\ \partial_n w_0^+ = -a_0\lambda\mu_\lambda. \end{cases} \qquad \begin{cases} w_0^- = b_0 + 2\log\mu_\lambda \\ \partial_n w_0^- = a_0\lambda\mu_\lambda. \end{cases}$$

(*) and (**) give one overdetermined, nonlinear problem for μ_λ.

The match

At this point we solve (*)+(**) only with a certain precision.
Hence we find

$$w_0^{\pm} \approx (\beta + \log \beta) W_\gamma^{\pm}$$

where

$$\beta := 2 \log \frac{1}{a_0 \lambda} + b_0 \qquad \mu_\lambda = -\frac{\beta + \log \beta}{a_0 \lambda} \partial_n W_\gamma^+ = \frac{\beta + \log \beta}{a_0 \lambda} \partial_n W_\gamma^-$$

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Then with this choice:

- $\partial_n w_0^\pm \pm a_0\lambda\mu_\lambda = O(1)$ on γ;

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- $\partial_n w_0^{\pm} \pm a_0 \lambda \mu_\lambda = O(1)$ on γ;
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A fixed point argument

We define the first approximation:

$$u_0 = \boxed{\chi_0 v_0} + \boxed{\chi_0^+ w_0^+} + \boxed{\chi_0^- w_0^-}$$

where χ_0 and $\chi_0^{\pm}$ are some cutoff functions overlapping at the distance $O(\delta_\lambda)$ of γ.

A fixed point argument

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where χ_0 and $\chi_0^\pm$ are some cutoff functions overlapping at the distance $O(\delta_\lambda)$ of γ.We want to set up a fixed point scheme to find a function $\psi \in H_0^1(\Omega)$ such that

$$\Delta(u_0 + \psi) + \lambda^2 e^{u_0+\psi} = 0$$

i.e.

$$\underbrace{\Delta\psi + \lambda^2 e^{u_0}\psi}_{\substack{\mathcal{L}_{u_0}(\psi) \\ \text{linear operator at } u_0}} = -\underbrace{(\Delta u_0 + \lambda^2 e^{u_0})}_{\substack{\mathcal{E}_{u_0} \\ \text{error}}} - \underbrace{\lambda^2 e^{u_0}(e^\psi - 1 - \psi)}_{\substack{\mathcal{Q}(\psi) \\ \text{quadratic in } \psi}}$$

The error

The error of the first order approximation close to the curve is still quite large and not even bounded!

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Indeed, we use stretched variables $\eta = \lambda \mu_\lambda t$ and we get

$$\mathcal{E}_{u_0} \sim \underbrace{\partial^2_{\eta\eta}\mathcal{U} + e^{\mathcal{U}}}_{:=0} + \log\frac{1}{\lambda}\mathcal{U}' = O\left(\log\frac{1}{\lambda}\right)$$

Improvement of the solution

We look for a solution of $(\mathcal{P}_\lambda)$ in the form

$$u = u_0 + u_1 + u_2$$

where

$$u_0 = \chi_0 v_0 + \chi_0^+ w_0^+ + \chi_0^- w_0^-$$

$$u_1 = \chi_1 \bar{v}_1 + \chi_1^+ w_1^+ + \chi_1^- w_1^-$$

$$u_2 = \chi_2 v_2$$

where

1. χ_0 and $\chi_0^\pm$ are some cutoff functions overlapping at the distance $O(\delta_\lambda)$ of γ;
2. χ_1 and $\chi_1^\pm$ are some cutoff functions overlapping at the distance $O(m_1\delta_\lambda)$ of γ with $m_1 > 2$;
3. χ_2 is a cutoff function supported in $|t| < 2m_2\delta_\lambda$ with $m_2 > m_1$

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$$u_2 = \chi_2 v_2$$

where

1. $\bar{v}_1$ is an "almost" affine function so that the error of $v_0 + \bar{v}_1$ is small in the inner region;
2. $w_1^{\pm}$ is an "almost" harmonic function in the domain (up to the curve) which matches with $\bar{v}_1$ close to the curve
3. v_2 is an exponentially decaying function

How to build $\bar{v}_1$: the linear theory close to the curve

In the inner region we let the linear operator

$$\boxed{\mathcal{L}_{u_0} = \partial_\eta^2 + e^{U(\eta)}}.$$

where $\eta = \lambda \mu_\lambda t$ is a stretched variable.
The fundamental set $\mathcal{K}$ of $\mathcal{L}_{u_0}$ is given by

$$\mathcal{K} = \text{span}\{\varphi_1, \varphi_2\} \tag{1}$$

where

$$\varphi_1 = U' = -2\frac{\eta}{|\eta|} + O(e^{-2|\eta|}) \quad \text{and} \quad \varphi_2 = \eta U' + 2 = -2|\eta| + 2 + O(e^{-2|\eta|})$$

$\bar{v}_1(s,\eta)$ solves

$$\begin{cases} \mathcal{L}_{u_0} v = g & \text{in } [0, \ell(\gamma)] \times \mathbb{R} \\ \displaystyle\int_{\mathbb{R}} g(\eta)\varphi_1(\eta)\, d\eta = 0 = \int_{\mathbb{R}} g(\eta)\varphi_2(\eta)\, d\eta. \end{cases}$$

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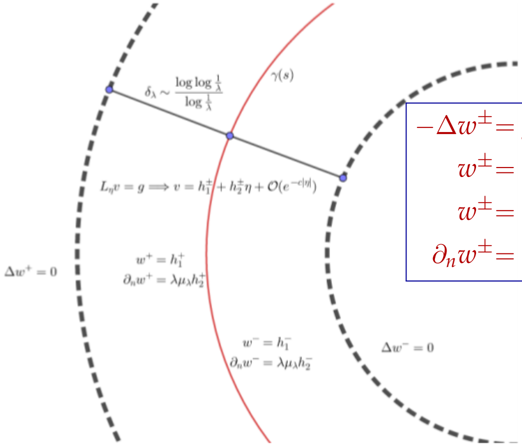
namely

$$\bar{v}_1(s,\eta) = \bar{h}_1(s)\varphi_1(\eta) + \bar{h}_2(s)\varphi_2(\eta) = \underbrace{\bar{h}_1^{\pm}(s)}_{\textit{free}} + \underbrace{\bar{h}_2^{\pm}(s)}_{\textit{free}}\,\eta + \tilde{v}_1$$

as $\eta \to \pm\infty$.

How to build $w_1^{\pm}$: the linear theory associated to the free boundary problem



$$\boxed{\begin{aligned} -\Delta w^{\pm} &= g^{\pm}, && \text{in } \Omega^{\pm}, \\ w^{\pm} &= 0, && \text{on } \partial\Omega^{\pm} \cap \partial\Omega. \\ w^{\pm} &= h_1^{\pm}, && \text{on } \gamma, \\ \partial_n w^{\pm} &= \lambda\mu_\lambda h_2^{\pm}, && \text{on } \gamma. \end{aligned}}$$

The right matching problem

We look for functions $w^{\pm}$ such that

$$\boxed{\begin{aligned} -\Delta w^{\pm} &= g^{\pm}, \qquad \text{in } \Omega^{\pm}, \\ w^{\pm} &= 0, \qquad \text{on } \partial\Omega^{\pm} \cap \partial\Omega. \\ w^{\pm} &= h_1^{\pm}, \quad \text{on } \gamma, \\ \partial_n w^{\pm} &= \lambda \mu_\lambda h_2^{\pm}, \quad \text{on } \gamma. \end{aligned}}$$

the matching problem is overdetermined and its solution are the functions $w^{\pm}$ together with the function h.

A result for solving the *matching problem*

Proposition

There exists a sequence $\lambda_n \to 0$ such that for any $g^{\pm} \in L^2(\Omega^{\pm})$ the matching problem has a solution $w^{\pm} \in H^2(\Omega^{\pm})$ and $h \in H^1(\gamma)$.

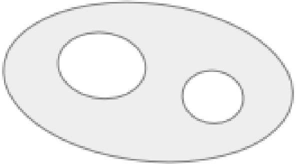
Other idea for solving the problem: suggested by S. Terracini

The function
$\mathcal{W} := \min\{u, 2-u\}$ and the
curve $\gamma := \{u = 1\}$
solves the problem

$$\begin{cases} \Delta \mathcal{W} = 0 & \text{in } \Omega \setminus \gamma \\ \mathcal{W} = 0 & \text{on } \partial\Omega \\ \mathcal{W} = 1 & \text{on } \gamma \\ \partial_{\nu^+}\mathcal{W} = \partial_{\nu^-}\mathcal{W} & \text{on } \gamma. \end{cases}$$

One can obtain the same result by making some changes (technical changes) in the proof and by means of the Dirichlet to Neumann map!

In a general multiply connected domain: work in progress..



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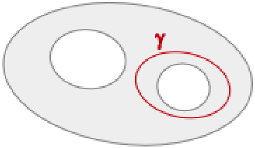

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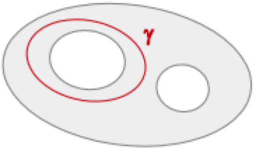

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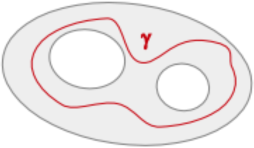

Related results

Let

$$(\mathcal{KS}) \qquad \begin{cases} \Delta u + u + \lambda e^u = 0 & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega \end{cases}$$

which comes up as a stationary equation for the Keller-Segel model.

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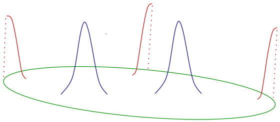
- $(\mathcal{KS})$ has bubbling solutions, including bubbles on $\partial\Omega$



del Pino, Wei *Nonlinearity (2006);*

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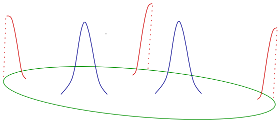

  del Pino, Wei *Nonlinearity (2006);*

- It has solutions that blow up along the whole boundary.

  del Pino, Pistoia, V. *J. of Diff. Eqs (2016);*
  Pistoia, V. *Proc. of the Royal Soc. of Ed. Sec A (2015);*

References

- M. Kowalczyk, A. Pistoia, G. Vaira,
  Maximal solution of the Liouville equation in doubly connected domain,
  Journal of Functional Analysis Vol. 277, Issue 9, Pages 2997-3050
- M. Kowalczyk, A. Pistoia, P. Rybka, G. Vaira,
  Free boundary problems arising in the theory of maximal solutions of equations with exponential nonlinearities,
  preprint
- M. Kowalczyk, A. Pistoia, G. Vaira, *Maximal solution of the Liouville equation in a not simply connected domain,*
  in preparation.

Thank you for the attention!