## Incontro su

Variational methods, with applications to problems in mathematical physics and geometry in occasione del 75 -simo compleanno di

Antonio Ambrosetti
M. Berti, V. Coti Zelati \& A. Malchiodi

Patrizia Pucci<br>Università degli Studi di Perugia

Unità locale di Perugia: alcuni risultati e prospettive di ricerca

Venezia, December 1st, $2019 \star$ 9:00-9:45


Antonio Ambrosetti

WAVE EQUATIONS WITH HYPERBOLIC DYNAMICAL BOUNDARY CONDITIONS AND SOURCES

- E. Vitillaro, Arch. Rat. Mech. Anal. 2017
- $" \quad$, J.D.E. 2018 $(P) \begin{cases}u_{t t}-\Delta u+\alpha\left|u_{t}\right|^{m-2} u_{t}=\gamma|u|^{p-2} u & \text { in }(0, \infty) \times \Omega, \\ u=0 & \text { on }(0, \infty) \times \Gamma_{0}, \\ u_{t t}+\partial_{\nu} u-\Delta_{\Gamma} u+\beta\left|u_{t}\right|^{\mu-2} u_{t}=\delta|u|^{q-2} u & \text { on }(0, \infty) \times \Gamma_{1}, \\ u(0, x)=u_{0}(x), \quad u_{t}(0, x)=u_{1}(x) & \text { in } \bar{\Omega}\end{cases}$
- $\Omega \subset \mathbb{R}^{N}, N \geq 2$, bounded, open $C^{1}, \Gamma:=\partial \Omega$;
- $\Gamma=\Gamma_{0} \cup \Gamma_{1}, \Gamma_{0} \cap \Gamma_{1}=\emptyset, \Gamma_{1} \neq \emptyset$ relatively open on $\Gamma, \sigma\left(\overline{\Gamma_{0}} \cap \overline{\Gamma_{1}}\right)=0, \sigma$ hypersurface measure on $\Gamma$,
- $u=u(t, x), t \geq 0, x \in \Omega, \Delta=\Delta_{x}, \Delta_{\Gamma}$

Laplace-Beltrami operator on the manifold $\Gamma$;

- $\nu=$ outward normal to $\Omega$;
- $m, \mu>1, \gamma, \delta \in \mathbb{R}, \alpha, \beta \geq 0, p, q \geq 2$.


## The TWO-DIMENSIONAL MODEL: A BASS DRUM


u vertical displacement of the membrane surface and linear mass densities are constant, normalized to 1
the tension of the membrane is constant, normalized to 1
possible conservative forces in the membrane and on the border
possible frictional damping terms in the membrane and on the internal border

Lagrangian function ( $\nabla_{\Gamma}$ Riemannian gradient):
$\mathcal{L}(u)=\frac{1}{2} \int_{\Omega}\left[u_{t}^{2}-|\nabla u|^{2}-\frac{2}{p}|u|^{p}\right]+\frac{1}{2} \int_{\Gamma_{1}}\left[u_{t}^{2}-\left|\nabla_{\Gamma} u\right|^{2}-\frac{2}{q}|u|^{q}\right]$

## References.

- Georgiev, Todorova
- Levine, P. P., Serrin
- Bociu, Lasiecka

9
9

- J. Graber, Said-Houari
- J. Graber
- Vitillaro
- Lasiecka, Fourrier
- Jameson Graber, Lasiecka
- Figotin, Reyes
- Zahn
J.D.E. '94

Contemp. Math. '97
Nonlinear Anal. '08
D.C.D.S. A '08
J.D.E. '10

Appl. Math. Optim. '12
J. Evol. Eqs. '12

Contemp. Math. '13
Evol. Equ. Control Theory '13
Semigroup Forum '14
J. Math. Phys. '15

Ann. I. H. P. Phys. Th. '17

Set
$r_{\Omega}=\left\{\begin{array}{ll}\frac{2 N}{N-2} & \text { if } N \geq 3, \\ \infty & \text { if } N=2,\end{array} \quad r_{\Gamma}= \begin{cases}\frac{2(N-1)}{N-3} & \text { if } N \geq 4, \\ \infty & \text { if } N=2,3 .\end{cases}\right.$

- $|u|^{p-2} u$ is subcritical when $p \leq 1+\frac{r_{\Omega}}{2}$,
supercritical when $1+\frac{r_{\Omega}}{2}<p \leq r_{\Omega}$,
super-supercritical when $p>r_{\Omega}$
- analogous classification for $|u|^{q-2} u$
- the subcritical case: both terms are subcritical
- the supercritical case: both terms are subcritical or supercritical, but for the previous case
- the super-supercritical case: the remaining case

$$
\begin{aligned}
& L^{2}\left(\Gamma_{1}\right)=\left\{u \in L^{2}(\Gamma): u=0 \text { a.e. on } \Gamma_{0}\right\} \\
& H^{0}=L^{2}(\Omega) \times L^{2}\left(\Gamma_{1}\right) \\
& H^{k}=\left\{(u, v) \in\left[H^{k}(\Omega) \times H^{k}(\Gamma)\right] \cap H^{0}: v=u_{\mid \Gamma}\right\}
\end{aligned}
$$

The subcritical case (Arch. Rat. Mech. Anal. 2017)
Theorem (Local Hadamard well-posedness in $\boldsymbol{H}^{1} \times \boldsymbol{H}^{0}$ )
Local existence, uniqueness, continuous dependence on data.

Proofs: nonlinear semigroups, estimates, monotonicity arguments.

Main difficulties:

- setting up the right pivot space since $\alpha, \beta$ can vanish or not;
- characterize generalized solutions as distributional ones.

ThEOREM (Regularity in $\boldsymbol{H}^{2} \times \boldsymbol{H}^{1}$ )

Technical difficulty (lack of references): regularity for the operator $-\Delta_{\Gamma}$ when $\Gamma$ is only $C^{2}$.

Theorem (Blow-up for linear damping terms)
Suppose that $m=\mu=2, p, q>2, \gamma, \delta \geq 0, \gamma+\delta>0$. Then for suitable data solutions blow-up in finite time. Assumption for global existence of solutions:
(GE) $\gamma \leq 0$ or $p=2$ or $[\alpha>0$ and $p \leq m]$, $\delta \leq 0$ or $q=2$ or $[\beta>0$ and $q \leq \mu]$

Theorem (Global existence and dynamical system generation)
Suppose that (GE) holds for
$(P) \begin{cases}u_{t t}-\Delta u+\alpha\left|u_{t}\right|^{m-2} u_{t}=\gamma|u|^{p-2} u & \text { in }(0, \infty) \times \Omega, \\ u=0 & \text { on }(0, \infty) \times \Gamma_{0}, \\ u_{t t}+\partial_{\nu} u-\Delta_{\Gamma} u+\beta\left|u_{t}\right|^{\mu-2} u_{t}=\delta|u|^{q-2} u & \text { on }(0, \infty) \times \Gamma_{1}, \\ u(0, x)=u_{0}(x), \quad u_{t}(0, x)=u_{1}(x) & \text { in } \bar{\Omega}\end{cases}$
Then all solutions are global in time. Consequently the problem generates a dynamical system in $H^{1} \times H^{0}$.

The non subcritical case (J. D. E. 2018)

- we are above the Serrin's hyperbolas:
$2 \leq p \leq 1+\frac{r_{\Omega}}{\bar{m}^{\prime}}, 2 \leq q \leq 1+\frac{r_{\Gamma}}{\bar{\mu}^{\prime}}(\bar{\rho}=\max \{\rho, 2\}) ;$
corresponding regions in relation with criticality:

- a supercritical source needs a corresponding effective damping:
$p>1+\frac{r_{\Omega}}{2} \Rightarrow \alpha>0, q>1+\frac{r_{\Gamma}}{2} \Rightarrow \beta>0 ;$
- dimension restrictions on the Serrin's hyperbolas: if $1+\frac{r_{\Omega}}{r^{2}}<p=1+r_{\Omega} / m^{\prime}$ then $N \leq 4$ if $1+\frac{r_{\Gamma}}{2}<q=1+r_{\Gamma} / \mu^{\prime}$ then $N \leq 5$;
- assumption for uniqueness and well-posedness: if $p>1+\frac{r_{\Omega}}{r^{2}}$ then $N \leq 4$; if $q>1+\frac{r_{\Gamma}}{2}$ then $N \leq 5$.

We summarize in the following table the extensions obtained from the subcritical case in dimensions $N=3$.
For simplicity $\alpha, \beta>0$ in
$(P) \begin{cases}u_{t t}-\Delta u+\alpha\left|u_{t}\right|^{m-2} u_{t}=\gamma|u|^{p-2} u & \text { in }(0, \infty) \times \Omega, \\ u=0 & \text { on }(0, \infty) \times \Gamma_{0}, \\ u_{t t}+\partial_{\nu} u-\Delta_{\Gamma} u+\beta\left|u_{t}\right|^{\mu-2} u_{t}=\delta|u|^{q-2} u & \text { on }(0, \infty) \times \Gamma_{1}, \\ u(0, x)=u_{0}(x), \quad u_{t}(0, x)=u_{1}(x) & \text { in } \bar{\Omega}\end{cases}$

## TABLE'S LEGENDA.

- One needs to take $u_{0}$ in a suitable space

$$
H^{1, \rho}=H^{1} \cap\left[L^{\rho}(\Omega) \times L^{2}\left(\Gamma_{1}\right)\right], \quad \rho>2
$$

while $u_{1} \in \boldsymbol{H}^{0}$.

- Since well-posedness yields existence-uniqueness, which in turn yields existence, when two or three results hold in the same space only the strongest result is explicitly written.
- All results are local in time and, if (GE) holds, also global in time. If the word "local" is added only the local version is available.
- When $N=3$ we have $H^{1}=H^{1,6}$.

Table 1. Main results when $N=3$

|  | $2 \leq q<\infty$ |
| :---: | :---: |
| $2 \leq p \leq 4$ | Subcritical and supercritical region |
| $4<p<6$ | well-posedness in $H^{1}$ |
| $6=p<m$ | existence-uniqueness in $H^{1}$ |
| Sobolev critical line | well-posedness in $H^{1,6+\varepsilon}$ |
| $6=p=m$ | existence-uniqueness in $H^{1}$ |
| $6<p<1+6 / m^{\prime}$ | local existence in $H^{1}$ |
|  | global existence in $H^{1, p}$ |
| Super-supercritical region, outside the Serrin hyberbola and on it | existence-uniqueness in $H^{1,3(p-2) / 2}$ |
|  | well-posedness in $H^{1,3(p-2) / 2+\varepsilon}$ |
| $6<p=1+6 / m^{\prime}$ | existence-uniqueness in $H^{1,3(p-2) / 2}$ |
|  | well-posedness in $H^{1,3(p-2) / 2+\varepsilon}$ |

Arguments in the proofs: truncations of Nemitskii operators, monotonicity argument to pass to the limit, nontrivial estimates, combination of compactness

Open problems, for simplicity in dimension $N=3$ :
$(P) \begin{cases}u_{t t}-\Delta u+\alpha\left|u_{t}\right|^{m-2} u_{t}=\gamma|u|^{p-2} u & \text { in }(0, \infty) \times \Omega, \\ u=0 & \text { on }(0, \infty) \times \Gamma_{0}, \\ u_{t t}+\partial_{\nu} u-\Delta_{\Gamma} u+\beta\left|u_{t}\right|^{\mu-2} u_{t}=\delta|u|^{q-2} u & \text { on }(0, \infty) \times \Gamma_{1}, \\ u(0, x)=u_{0}(x), \quad u_{t}(0, x)=u_{1}(x) & \text { in } \bar{\Omega}\end{cases}$

- to get Hadamard well-posedness in the energy space $H^{1} \times H^{0}$ in the non-subcritical ranges;
- to get Hadamard well-posedness in any space when $p=m=6$;
- (old standing problem) to skip the main Serrin's assumption

$$
2 \leq p \leq 1+\frac{r_{\Omega}}{\bar{m}^{\prime}}, \quad 2 \leq q \leq 1+\frac{r_{\Gamma}}{{\overline{\mu^{\prime}}}^{\prime}}
$$

## Carleman estimates and null controllability

Carleman estimates relate weighted Sobolev norms of the solution to some differential problem with weighted Lebesgue norms of the right-hand-side. For instance, if

$$
u_{t}-\Delta u=f \chi_{\omega} \quad \text { in } Q_{T}=\Omega \times(0, T), \quad \omega \subset \subset \Omega,
$$

take $\theta(t)=t^{-1}(T-t)^{-1}$ and a suitable weight $\phi(x, t)<0$. Then there exist $C>0, s_{0}>0$ such that for all $s \geq s_{0}$
$\int_{Q_{T}}\left(s \theta(t)|D u|^{2}+s^{3} \theta^{3}(t) u^{2}\right) e^{2 s \phi} d x d t \leq C \int_{Q_{T}} f^{2} e^{2 s \phi} d x d t+B . T$.
Main applications: unique continuation properties and observability inequalities for solutions of the homogeneous adjoint problem

$$
v_{t}+\Delta v=0 \quad \text { in } \Omega \times(0, T),
$$

namely

$$
\int_{\Omega}|v(0)|^{2} d x \leq C \int_{0}^{T} \int_{\omega}|v|^{2} d x d t
$$

## Carleman estimates and null controllability

The main consequence of the observability inequality is the null controllability for the original problem

$$
u_{t}-\Delta u=f \chi_{\omega} \quad \text { in } Q_{T}=\Omega \times(0, T), \quad \omega \subset \subset \Omega:
$$

there exists a localized control $f \in L^{2}(\omega \times(0, T))$ such that the solution of

$$
u_{t}-\Delta u=f \chi_{\omega} \quad \text { in } Q_{T}
$$

satisfies $u(x, T)=0$ for all $x \in \Omega$.

## Carleman estimates and null controllability

In G. Fragnelli, D. Mugnai, Memoirs AMS (2016) there are the first Carleman estimates for degenerate parabolic equations with interior non smooth (not $C^{1}$ ) degeneracy, leading to null controllability results for

$$
u_{t}-\left(a(x) u_{x}\right)_{x}=f \chi_{\omega} \quad \text { in }(0,1) \times(0, T)
$$

In G. Fragnelli, D. Mugnai, Adv. Nonlinear Anal. (2017) these results are extended to singular equations like

$$
u_{t}-\left(a(x) u_{x}\right)_{x}-\lambda \frac{u}{b(x)}=f \chi_{\omega} \quad \text { in }(0,1) \times(0, T)
$$

Final result: $\exists f \in L^{2}(\omega \times(0, T))$ s.t. $u(T)=0$ in $(0,1)$. Here $a(x) \sim\left|x-x_{0}\right|^{\alpha}, b(x) \sim\left|x-x_{0}\right|^{\beta}, x_{0} \in(0,1)$, $\alpha, \beta \in(0,2)$.
Two difficulties: the degeneracy point is in the interior and $a$ and $b$ are non smooth: no previous result can be applied!

## Carleman estimates and null controllability

Extensions or applications in
G. Fragnelli, D. Mugnai, EJQTDE (2018),
G. Fragnelli, D. Mugnai, Opuscula Math. (2019),
R. Du, J. Eichhorn, Q. Liu, C. Wang, ANonA (2019),
P. Cannarsa, R. Ferretti, P. Martinez, SICON (2019),
G. Fragnelli, D. Mugnai, $D C D S-S$ (2019).

The results are analogous to the smooth or non degenerate cases in
J.L. Vázquez, E. Zuazua, J. Funct. Anal. (2000),
S. Micu, E. Zuazua, Trans. Amer. Math. Soc. (2001),
F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli, J.

Evol. Eqs (2006),
J. Vancostenoble, E. Zuazua, J. Funct. Anal. (2008),
J. Vancostenoble, E. Zuazua, SIA M J. Math. Anal. (2009).

In the non smooth interior degenerate case, new techniques and ideas are needed.

BIG open problem: $N$-dimensional case.

## Fujita type results

- R. Filippucci, S. Lombardi, J. Diff. Equations (2019).
consider nonexistence of solutions for the prototype
(1) $\begin{cases}u_{t}-\Delta_{p} u \geq a(x) u^{q}-b(x) u^{m}|D u|^{s} & \text { in } \mathbb{R}^{N} \times \mathbb{R}^{+} \\ u(x, 0)=u_{0}(x) \geq 0, & \text { in } \mathbb{R}^{N},\end{cases}$
where $a(x), b(x)$ are nonnegative weights possibly singular or degenerate at $x=0$, while

$$
1<p<N, \quad 0 \leq m<q, \quad s \in\left(0, \frac{p(q-m)}{q+1}\right]
$$

PROBLEM: Is there a Fujita critical exponent $q_{F}$ such that if $1<q \leq q_{F}$ then (1) does not admit nontrivial nonnegative solutions?

- Fujita J. Fac. Sci. Univ. Tokyo (1966) $q_{F}=1+\frac{2}{N}$ for

$$
\begin{cases}u_{t}-\Delta u=u^{q} & \text { in } \mathbb{R}^{N} \times(0, \infty) \\ u(x, 0)=u_{0}(x) \geq 0 & \text { in } \mathbb{R}^{N} .\end{cases}
$$

Motivation for the gradient case:

$$
\begin{cases}u_{t}-\Delta u=u^{q}-b u^{m}|D u|^{s} & \text { in } \mathbb{R}^{N} \times \mathbb{R}^{+} \\ u(x, 0)=u_{0}(x) \geq 0, s, b>0, m \geq 0, & \text { in } \mathbb{R}^{N}\end{cases}
$$

model for the evolution of a biological species where $-b u^{m}|D u|^{s}$ describes the greed of the predator stimulated by displacements of preys and their concentration.

- Souplet, Weissler J. Math. Anal. Appl. (1997) m=0
- Souplet, Weissler Ann. Inst. H. Poincare, (1999) $m=0$
- Chlebik, Fila Rend. Mat. Appl. (1999) $m=0$
- Bartier Asymptot. Anal. (2006) $m \geq 0$
- Mitidieri, Pohozaev Dokl. Mat. (2002) consider

$$
\begin{cases}u_{t}-\Delta_{p} u \geq u^{q}-b|D u|^{p q /(q+1)} & \text { in } \mathbb{R}^{N} \times \mathbb{R}^{+} \\ u(x, 0)=u_{0}(x) \geq 0, & \text { in } \mathbb{R}^{N}\end{cases}
$$

and for $b>0$ small find $q_{F}=p-1+p / N$.

In [R. Filippucci, S. Lombardi, J. Diff. Equations (2019)] for the prototype problem
$\begin{cases}u_{t}-\Delta_{p} u \geq a_{0}|x|^{-\alpha} u^{q}-b_{0}|x|^{-\beta} u^{m}|D u|^{s} & \text { in } \mathbb{R}^{N} \backslash\{0\} \times \mathbb{R}^{+} \\ u(x, 0)=u_{0}(x) \geq 0, \quad 0 \leq m<q & \text { in } \mathbb{R}^{N},\end{cases}$
we have $q_{F}=p-1+p / N$ if $s=p(q-m) /(q+1)$,
$\alpha=\beta=0$ and $b_{0}$ small.
If $s<p(q-m) /(q+1)$ we have that $q_{F}$ can assume two values

$$
\begin{aligned}
& q_{F, 1}=p-1-\frac{\alpha}{N}+\frac{p}{N} \\
& q_{F, 2}=p-1-\frac{\alpha}{N}+\frac{p(N \Lambda+\beta)}{N(p-s)}, \quad \Lambda=s+m-p+1
\end{aligned}
$$

$$
\begin{aligned}
& \text { In particular } \\
& \quad q_{F, 1} \leq q_{F, 2} \Leftrightarrow s \geq p-\frac{N(m+1)-\beta}{N+1} \Leftrightarrow \beta \geq p-s-N \Lambda .
\end{aligned}
$$

Technique used: a priori estimates, with no conditions on the behavior of the solution at infinity, and the method of nonlinear capacity both adapted to our model containing several parameters.

- R. Filippucci, C. Lini, Discrete Contin. Dyn. Syst., S (2019)

Existence of positive weak solutions of

$$
\begin{cases}-\Delta_{p} u=f(x, u, D u), & \text { in } \Omega \\ u(x)=0, & \text { on } \partial \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{N}$, is a smooth BOUNDED domain, $1<p<N, f \in C\left(\Omega \times \mathbb{R}_{0}^{+} \times \mathbb{R}^{N} ; \mathbb{R}_{0}^{+}\right)$and there exist $c_{0} \geq 1$, $M>0$ s.t. in $\Omega \times \mathbb{R}_{0}^{+} \times \mathbb{R}^{N}$ holds
(F) $\quad u^{q}-M u^{m}|\eta|^{s} \leq f(x, u, \eta) \leq c_{0} u^{q}+M u^{m}|\eta|^{s}$

$$
\begin{gathered}
q \in\left(\boldsymbol{p}-\mathbf{1}, \boldsymbol{p}_{*}-\mathbf{1}\right) \quad \boldsymbol{p}_{*}=\frac{\boldsymbol{p}(\mathbf{N}-\mathbf{1})}{\boldsymbol{N}-\boldsymbol{p}} \\
\boldsymbol{m} \in[\mathbf{0}, \boldsymbol{K}), \quad \boldsymbol{K}=\min \left\{\boldsymbol{p - 1}, \frac{\boldsymbol{q}}{\boldsymbol{N}}, \boldsymbol{q}-\frac{\boldsymbol{s}}{\boldsymbol{p}}(\boldsymbol{q}+\mathbf{1})\right\} . \\
s \in\left(p-m-1, \frac{p(q-m)}{q+1}\right)
\end{gathered}
$$

- Ruiz, J. Diff. Equations (2004) $m=0$ in (F)

Ruiz produces a slight modification of the blow-up technique of Gidas and Spruck, based mainly on a scaling argument and on the use of a Liouville theorem applied to the limit problem $-\Delta_{p} u \geq C u^{q}$ both in the entire $\mathbb{R}^{N}$ (if a certain sequence converges to $x_{0} \in \Omega$ ) and in the halfspace $\mathbb{R}_{+}^{N}$ (if $x_{0} \in \partial \Omega$ ). The latter case is very delicate because Liouville type results for the $p$-Laplacian in the halfspace are not available in a general setting. Thus, he uses a consequence of Harnack's inequality

- N. Trudinger, Comm. Pures Appl. Math. (1967)
to avoid the case of the halfspace and thus the range in which the exponent $q$ can vary is smaller, $q \in\left(p-1, p_{*}-1\right)$ instead of $q \in\left(p-1, p^{*}-1\right)$.
Main difficulties: produce a priori integral estimates for nonnegative solutions of

$$
-\Delta_{p} u \geq u^{q}-M u^{m}|D u|^{s}, \quad \text { in } \Omega \subset \mathbb{R}^{N}
$$

as in

- J. Serrin, Acta Math. 1964.
- L. Baldelli, R. Filippucci Discrete Contin. Dyn. Syst. - S (2019)

Problem: Prove pointwise a priori estimates of the form

$$
u(x)+|D u(x)|^{\mu_{1}} \leq C\left(1+d i s t^{-\mu_{2}}(x, \partial \Omega)\right)
$$

with $\mu_{1}, \mu_{2}>0$ depending on the parameters of the equation, for positive solutions of

$$
-\Delta_{p} u=f(x, u, D u) \quad \text { in } \Omega
$$

where $\Omega \subseteq \mathbb{R}^{N}$ is a domain, $1<p<N$ and $f: \Omega \times \mathbb{R}_{0}^{+} \times \mathbb{R}^{N} \rightarrow \mathbb{R}$ is Caratheodory function whose main prototype is

$$
\begin{gathered}
f=a(x) u^{q}-b(x) u^{m}|D u|^{s} \\
\boldsymbol{q}, \boldsymbol{s}>\mathbf{0}, \boldsymbol{m} \geq \mathbf{0}, \mathbf{0}<\boldsymbol{a}(\boldsymbol{x}) \in \boldsymbol{C}(\overline{\boldsymbol{\Omega}}) \text { and } \boldsymbol{b}(\boldsymbol{x}) \in \boldsymbol{L}^{\infty}(\boldsymbol{\Omega}) .
\end{gathered}
$$

- Polacik, Quitter, Souplet, Duke Math. J. (2007)
- Serrin, Zou, Acta Math. (2002) called universal a priori estimates


## Technique used:

- Scaling as in the blowup technique;
- Gidas e Spruck Comm. Partial Diff. Eq. (1981)
- Doubling Lemma (which allows us to avoid the case of the halfspace);
- Polacik, Quitter, Souplet, Duke Math. J. (2007)
- Hu, Diff. Int. Eq. (1996)
- Liouville theorem for the limit problem.
- Mitidieri e Pohozaev Dokl. Akad. Nauk (1998)
- Serrin e Zou Acta Math. (2002)

Main difficulties: to apply the scaling argument to a nonlinearity depending on the gradient, indeed Polacik, Quitter and Souplet considered the case $p=2$ with $f=f(x, u, D u)$ and the case $p \neq 2$ with $f_{b}=f(x, \underline{\underline{\underline{E}}})$.

Open problems:
(a) Pointwise a priori estimates for $-\Delta_{p} u=a(x) u^{q}|D u|^{s}$ in $\Omega$, with $q, \theta>0$ and $0<a(x) \in C(\bar{\Omega})$. The main difficulty is due to the fact that the limit problem is $-\Delta_{p} u=u^{q}|D u|^{s}$ in $\mathbb{R}^{N}$ which admits all constants as entire solutions.
For Liouville type results for $-\Delta_{p} u \geq u^{q}|D u|^{s}$ we refer to

- Filippucci, J. Diff. Equations (2011) (in the inequality case $0<s \leq N(p-1) /(N-p), N>p)$.

For $-\Delta_{p} u=u^{q}|D u|^{s}$ the situation is very delicate and few things are known. In the case $p=2$ Liouville theorems are proved in

- Bidaut-Véron, García-Huidobro, Véron Duke Math. J. (2019) $0<s \leq 2$
- Filippucci, P. P., Souplet Adv. Nonlinear Stud. to appear $s>2$ and bounded solutions.
(b) cover the entire interval $q \in\left(p-1, p^{*}-1\right)$ for the non existence result of the Dirichlet problem with $f=a(x) u^{q}-b(x) u^{m}|D u|^{s}$ on bounded domains.


## A Gidas-Ni-Nirenberg Result

In the celebrated paper

- B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979), 209-243. the authors proved that any positive solution of the elliptic Euclidean equation

$$
-\Delta u=|u|^{\frac{4}{d-2}} u, \quad u \in C^{2}\left(\mathbb{R}^{d}\right), \quad d \geq 3
$$

which has finite energy is of the form

$$
u(x)=\left(\frac{\sqrt{d(d-2)} a}{a^{2}+|x-\xi|^{2}}\right)^{\frac{d-2}{2}}
$$

for some $a>0$ and $\xi \in \mathbb{R}^{d}$.

Thereafter, some people tried to show without success that all the solutions of above the problem, which are positive somewhere, have the same expression. Their efforts have to be in vain, as we will see shortly that the problem actually has a lot of solutions other than those given by Gidas, Ni and Nirenberg.
For instance, in

- W.Y. Ding, On a conformally invariant elliptic equation on $\mathbb{R}^{n}$, Comm. Math. Phys. 107 (1986), 331-335.
the author proved the following result


## Theorem

For $d \geq 3$, the equation $-\Delta u=|u|^{\frac{4}{d-2}} u$ on $\mathbb{R}^{d}$ has infinitely many distinct solutions with finite energy and which change sign.

Inspired by Ding,

- T. Bartsch, M. Schneider, T. Weth, Multiple solutions of a critical polyharmonic equation, J. Reine Angew. Math. 571 (2004), 131-143.
prove for the critical polyharmonic equation

$$
\left\{\begin{array}{l}
(-\Delta)^{m} u=|u|^{2_{m}^{*}-2} u \quad \text { in } \mathbb{R}^{d}, \quad u \in \mathcal{D}^{m, 2}\left(\mathbb{R}^{d}\right) \\
d>2 m, \quad 2_{m}^{*}=\frac{2 d}{d-2 m}
\end{array}\right.
$$

the existence of a sequence of infinitely many finite energy nodal solutions which are unbounded in the Sobolev space $\mathcal{D}^{m, 2}\left(\mathbb{R}^{d}\right)$.

Encouraged by a wide interest on the current literature on polyharmonic problems, G. Molica Bisci, P. P. prove that the above critical polyharmonic problem

$$
\left\{\begin{array}{l}
(-\Delta)^{m} u=|u|^{2_{m}^{*}-2} u \quad \text { in } \mathbb{R}^{d}, \quad u \in \mathcal{D}^{m, 2}\left(\mathbb{R}^{d}\right) \\
d>2 m, \quad 2_{m}^{*}=\frac{2 d}{d-2 m}
\end{array}\right.
$$

admits at least a finite number $\zeta_{d}$ of sequences of infinitely many finite energy nodal solutions which are unbounded in the Sobolev space $\mathcal{D}^{m, 2}\left(\mathbb{R}^{d}\right)$. Here the number $\zeta_{d}$ depends on the unrestricted partition function of $d$.

According to the celebrated asymptotic formulas due to Hardy, Ramanujan and Rademacher we obtain an exponential growth of the numbers of sequences of solutions for the critical polyharmonic problem.


The main approach is based on some arguments contained in:

- G. Molica Bisci, A group-theoretical approach for nonlinear Schrödinger equations, Advances in Calculus of Variations (in press).
- G. Molica Bisci, P. P., Multiple sequences of entire solutions for critical polyharmonic equations, Riv. Math. Univ. Parma (N.S.) 10, 117-144, (2019).
- G. Molica Bisci, P. P., Nonlinear Problems with Lack of Compactness, to appear in De Gruyter Series in Nonlinear Analysis and Applications, pages 340 (2020).
For related problems on the Riemannian case we refer to the papers
- G. Molica Bisci, L. Vilasi, Existence results for some problems on Riemannian manifolds, Comm. Anal. Geom. (in press).
- B. Bianchini, L. Mari, P. P., M. Rigoli, On the interplay among weak and strong maximum principles, compact support principles and Keller-Osserman conditions on manifolds, submitted for publication, pages 238.
Recently,
- A. Maalaoui, V. Martino, Changing-sign solutions for the CR-Yamabe equation, Differential Integral Equations 25, 601-609 (2012).
- A. Maalaoui, V. Martino, G. Tralli, Complex group actions on the sphere and sign changing solutions for the CR-Yamabe equation, J. Math. Anal. Appl. 431, 126-135 (2015).
motivated again by the original paper of Ding, establish the existence of changing sign solutions for the Yamabe problem on the Heisenberg group $\mathbb{H}^{d}$.

By using a similar theoretical algebraic tool, in the forthcoming book with G. Molica Bisci we study the existence of at least one solution for subelliptic equations defined on a special class of domains (possibly unbounded) of the Heisenberg group $\mathbb{H}^{d}=\mathbb{C}^{d} \times \mathbb{R}, \boldsymbol{d} \geq 1$. While other results for problems with lack of compactness are obtained in

- P.P., Critical Schrödinger-Hardy systems in the Heisenberg group, Discrete Contin. Dyn. Syst. Ser. S, Special Issue on the occasion of the 60th birthday of Professor Vicentiu D. Radulescu, 12 (2019), 375-400.
- P.P., Existence and multiplicity results for quasilinear elliptic equations in the Heisenberg group, Opuscula Math. 39 (2019), 247-257.
- P.P., Existence of entire solutions for quasilinear elliptic equations in the Heisenberg group, Minimax Theory Appl., Special Issue on Nonlinear Phenomena: Theory and Applications 4 (2019), 161-188.
- P. P., L. Temperini, Existence for ( $p, q$ ) critical systems in the Heisenberg group, Adv. Nonlinear Anal. 9 (2020), 895-922.
- S. Bordoni, R. Filippucci, P. P., Existence of solutions in problems on Heisenberg groups involving Hardy and critical terms, J. Geom. Anal., Special Issue Perspectives of Geometric Analysis in PDEs, pages 29.

In

- G. Molica Bisci, D. Repovš, Gradient-type systems on unbounded domains of the Heisenberg group, J. Geom. Anal., Special Issue Perspectives of Geometric Analysis in PDEs.
exploiting the abstract framework developed in the book, the following singular subelliptic system has been studied:

$$
\begin{cases}-\Delta_{\mathbb{H}^{d}} u-\nu V(q) u+u=\lambda K(q) \partial_{u} F(u, v) & \text { in } \Omega_{\psi} \\ -\Delta_{\mathbb{H} d} v-\nu V(q) v+v=\lambda K(q) \partial_{v} F(u, v) & \text { in } \Omega_{\psi} \\ u=v=0 \text { on } \partial \Omega_{\psi}, & \end{cases}
$$

where $\Omega_{\psi}$ is an unbounded strip of $\mathbb{H}^{d}$.


Figure 1. A simple prototype of $\Omega_{\psi}$

We notice that in our setting the strip $\boldsymbol{\Omega}_{\psi}$ is possible not strongly asymptotically contractive.

A general multiplicity existence theorem of changing sign solutions for the fractional Yamabe problem on the Heisenberg group $\mathbb{H}^{d}$ has been studied in

- A. Kristály, Nodal solutions for the fractional Yamabe problem on Heisenberg groups, Proc. Roy. Soc. Edinburgh Sect. A (in press).
via a nonlocal version of the Ding-Hebey-Vaugon compactness result on the Cauchy-Riemann unit sphere $\mathbb{S}^{2 d+1}$ and an algebraic theoretical approach on suitable subgroups of the unitary group $U(d+1)$.
- G. Molica Bisci, V. Rădulescu, R. Servadei, Variational methods for nonlocal fractional problems, Encyclopedia of Mathematics and its Applications, 162, 2016, pp. 400.
- G. Molica Bisci, Variational and Topological Methods for Nonlocal Fractional Periodic Equations, Recent developments in the Nonlocal Theory, Ed. by T. Kuusi and G. Palatucci, De Gruyter, 2018, 359-432.

In particular in

- G. Molica Bisci, J. Mawhin, A Brezis-Nirenberg type result for a nonlocal fractional operator, J. Lond. Math. Soc. 95 (2017), 73-93.
the perturbed Brezis-Nirenberg problem has been studied. This subject is also the aim of some recent papers.
- G.M. Figueiredo, G. Molica Bisci, R. Servadei, The effect of the domain topology on the number of solutions of fractional Laplace problems, Calc. Var. Partial Differential Equations 57 (2018) Art. 103, 24 pp.

In this paper the authors studied the multiplicity of solutions for the following fractional Laplace problem

$$
\begin{cases}(-\Delta)^{s} u=\mu|u|^{q-2} u+|u|^{2 *}-2 & \text { in } \Omega \\ u=0 & \text { in } \mathbb{R}^{n} \backslash \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{n}$ is an open bounded set with continuous boundary, $n>2 s$ with $s \in(0,1),(-\Delta)^{s}$ is the fractional Laplacian operator, $\mu$ is a positive real parameter, $q \in\left[2,2_{s}^{*}\right)$ and $2_{s}^{*}=2 n /(n-2 s)$ is the fractional critical Sobolev exponent. Using the Lusternik-Schnirelman theory, the number of nontrivial solutions of the problem under consideration is related with the topology of $\Omega$.

- A. Fiscella, G. Molica Bisci, R. Servadei, Multiplicity results for fractional Laplace problems with critical growth, Manuscripta Math. 155 (2018), 369-388.

This paper deals with multiplicity and bifurcation results for the following fractional critical problem

$$
\begin{cases}(-\Delta)^{s} u=\gamma|u|^{2_{s}^{*}-2} u+f(x, u) & \text { in } \Omega \\ u=0 & \text { in } \mathbb{R}^{n} \backslash \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{n}$ is an open bounded set with continuous boundary, $n>2 s$ with $s \in(0,1),(-\Delta)^{s}$ is the fractional Laplacian operator, $\gamma$ is a positive real parameter, $2_{s}^{*}=2 n /(n-2 s)$ is the fractional critical Sobolev exponent and $f$ is a suitable Carathéodory function satisfying different subcritical conditions.

For this problem the authors proved two different results of multiple solutions in the case when $f$ is an odd function. When $f$ has not any symmetry it is still possible to get a multiplicity result proving that the problem under consideration admits at least two solutions of different sign.

PERSPECTIVES: Get multiplicity results for elliptic problems with competitive nonlinearities, settled on noncompact manifolds in presence of a compact topological group action.
D. Mugnai, E. Proietti Lippi, Nonlinear Anal. (2019):

$$
\left\{\begin{array}{l}
(-\Delta)_{p}^{s} u=f(x, u) \quad \text { in } \Omega \\
\mathscr{N}_{s, p} u=g \quad \text { in } \mathbb{R}^{N} \backslash \bar{\Omega}
\end{array}\right.
$$

where

$$
(-\Delta)_{p}^{s} u(x)=P V \int_{\mathbb{R}^{N}}|u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+p s}} d y
$$

is the fractional $p$-Laplacian and
$\mathscr{N}_{s, p} u(x):=\int_{\Omega}|u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+p s}} d y, x \in \mathbb{R}^{N} \backslash \bar{\Omega}$,
is the nonlocal normal $p$-derivative, introduced for $p=2$ in
Di Pierro, Ros-Oton, Valdinoci, Rev. Mat. Iberoam. 2017, and for $p \in(1, \infty)$ in
Barrios, Montoro, Peral, Soria, Nonlinear Anal. 2018.

In D. Mugnai, E. Proietti Lippi, Nonlinear Anal. (2019) prove

## Theorem (Locality Theorem)

Let $\boldsymbol{u}$ be a weak solution of

$$
\left\{\begin{array}{l}
(-\Delta)_{p}^{s} u=f(x, u) \quad \text { in } \Omega \\
\mathscr{N}_{s, p} u=g \quad \text { in } \mathbb{R}^{N} \backslash \bar{\Omega}
\end{array}\right.
$$

Then $\mathscr{N}_{s, p} u=g$ a.e. in $\mathbb{R}^{N} \backslash \bar{\Omega}$.
Proposition (Maximum Principle or Rigidity Theorem)
Let $\boldsymbol{u}$ be a solution of

$$
\left\{\begin{array}{l}
(-\Delta)_{p}^{s} u=h(x) \quad \text { in } \Omega \\
\mathscr{N}_{s, p} u=g \quad \text { in } \mathbb{R}^{N} \backslash \bar{\Omega}
\end{array}\right.
$$

with $h \in L^{p^{\prime}}(\Omega), g \in L^{1}\left(\mathbb{R}^{N} \backslash \bar{\Omega}\right)$ with $h \geq 0$ and $g \geq 0$. Then, $u$ is constant in $\mathbb{R}^{N}$.
( $\boldsymbol{E}$ )

$$
\left\{\begin{array}{l}
(-\Delta)_{p}^{s} u=\lambda|u|^{p-2} u \quad \text { in } \Omega \\
\mathscr{N}_{s, p} u=0 \quad \text { in } \mathbb{R}^{N} \backslash \bar{\Omega}
\end{array}\right.
$$

## Proposition

There is a sequence of eigenvalues such that
$0=\lambda_{1}<\lambda_{2} \leq \ldots \leq \lambda_{k} \rightarrow \infty$.
Proposition
$u$ solution to $(E)$ with $u>0$ in $\Omega$. Then $\lambda=0$, so
$u=$ constant.
Proof not standard due to the non-locality.
Eigenfunctions associated to higher eigenvalues change $\operatorname{sign}$ in $\Omega$.
Theorem
$u$ solution of $(E)$ with $\lambda \geq 0$. Then $u \in L^{\infty}\left(\mathbb{R}^{N}\right)$ and $\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}=\|u\|_{L^{\infty}(\Omega)}$.
Proof: ideas from Franzina, Palatucci (2014) + Locality Theorem.

$$
\begin{cases}(-\Delta)_{p}^{s} u+|u|^{p-2} u=f(x, u) & \text { in } \Omega \\ \mathscr{N}_{s, p} u=0 & \text { in } \mathbb{R}^{N} \backslash \bar{\Omega}\end{cases}
$$

$f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a Carathéodory function such that $f(x, 0)=0$ for almost every $x \in \Omega$. Moreover:
$f$ has subcritical growth and doesn't satisfy the Ambrosetti-Rabinowitz condition. A new general condition is needed. It was introduced in Mugnai - Papageorgiou in Trans. AMS 2014.
In this way, one can prove
Theorem
Problem

$$
\begin{cases}(-\Delta)_{p}^{s} u+|u|^{p-2} u=f(x, u) & \text { in } \Omega \\ \mathscr{N}_{s, p} u=0 & \text { in } \mathbb{R}^{N} \backslash \bar{\Omega}\end{cases}
$$

has two non-trivial constant sign solutions.

## Some open questions

- Let u solve

$$
\left\{\begin{array}{l}
(-\Delta)_{p}^{s} u=f(x, u) \quad \text { in } \Omega  \tag{P}\\
\mathscr{N}_{s, p} u=g \quad \text { in } \mathbb{R}^{N} \backslash \bar{\Omega}
\end{array}\right.
$$

Is is true that $u$ is continuous in $\mathbb{R}^{N}$ ? In the Dirichlet case $u=0$ in $\mathbb{R}^{N} \backslash \bar{\Omega}$, this last condition helps significantly in obtaining the desired regularity.

- Is it true that any solution of problem (P) solves the equation $(-\Delta)_{p}^{s} u=f(x, u)$ pointwise in $\Omega$ ?

In A. Fiscella, P. P., $(p, N)$ equations with critical exponential nonlinearities in $\mathbb{R}^{N}$, J. Math. Anal. Appl., Special Issue New Developments in Nonuniformly Elliptic and Nonstandard Growth Problems, we prove existence and multiplicity results for

$$
(\mathcal{E}) \begin{aligned}
-\Delta_{p} u-\Delta_{N} u & +|u|^{p-2} u+|u|^{N-2} u \\
& =\lambda h(x) u_{+}^{q-1}+\gamma f(x, u) \quad \text { in } \mathbb{R}^{N}
\end{aligned}
$$

where $1<p<N<\infty, N \geq 2,1<q<N$, $u_{+}=\max \{u, 0\}$, and $h$ is a positive function of class $L^{\theta}\left(\mathbb{R}^{N}\right)$, with $\theta=N /(N-q)$, while $\lambda$ and $\gamma$ are positive parameters. The function $f$ is of exponential type and is assumed to satisfy very general restrictions. Of course we cover the classical special examples as

$$
f(u)=u_{+}\left(e^{u^{2}}-1\right), \quad u \in \mathbb{R}, \quad N=2
$$

and the most interesting case $N>2$

$$
\begin{gathered}
f(u)=u_{+}^{N-1}\left(e^{u_{+}^{N^{\prime}}}-S_{N-2}\left(1, u_{+}\right)\right), \\
S_{N-2}\left(1, u_{+}\right)=\sum_{j=0}^{N-2} \frac{u^{j N^{\prime}}}{j!}, N^{\prime}=\frac{N}{N-1} .
\end{gathered}
$$

The main features and novelty of the paper are the $(p, N)$ growth of the elliptic operator, combined with the double lack of compactness. In order to state a multiplicity result, we introduce a tricky step analysis based on the application of a completely new Brézis and Lieb type lemma for exponential nonlinearities. There are very few contributions devoted to the study of exponential nonlinear problems driven by operators with non-standard growth conditions. A ( $p, N$ ) equation similar to $(\mathcal{E})$ first appears in

- Y. Yang, K. Perera, ( $\boldsymbol{N}, \boldsymbol{q}$ )-Laplacian problems with critical Trudinger-Moser nonlinearities, Bull. London Math. Soc. 48 (2016), 260-270.
but set on a bounded domain $\Omega$ and with $f$ exactly equal to an exponential function. The authors are able to get an existence result via a suitable minimax argument, which strongly relies on the requirement that $\Omega$ is bounded. More recently, the existence of one solution for critical exponential problems, set on bounded domains $\Omega$ and driven by a general $(p, N)$ operator, is given in
- G.M. Figueiredo, F.B.M. Nunes, Existence of positive solutions for a class of quasilinear elliptic problems with exponential growth via the Nehari manifold method, Rev. Mat. Complut. 32 (2019), 1-18.
via the Nehari manifold approach.
While, for equations in the entire space, involving elliptic operators with standard $N$-growth as well as critical Trudinger-Moser nonlinearities, we refer to
- C.O. Alves, L.R. de Freitas, S.H.M. Soares, Indefinite quasilinear elliptic equations in exterior domains with exponential critical growth, Differential Integral Equations 24 (2011), 1047-1062.
- C.O. Alves, G.M. Figueiredo, Existence and multiplicity of positive solutions to a $p$-Laplacian equation in $\mathbb{R}^{N}$, Differential Integral Equations 19 (2006) 143-162.
- C.O. Alves, G.M. Figueiredo, On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^{N}$, J. Differential Equations 246 (2009), 1288-1311.
- J.M. do Ó, $N$-Laplacian equations in $\mathbb{R}^{N}$ with critical growth, Abstr. Appl. Anal. 2 (1997), 301-315.
for existence results, to
- C.O. Alves, L.R. de Freitas, Multiplicity results for a class of quasilinear equations with exponential critical growth, Topol. Methods Nonlinear Anal. 39 (2012), 222-244.
- C.O. Alves, L.R. de Freitas, Multiplicity of nonradial solutions for a class of quasilinear equations on annulus with exponential critical growth, Math. Nachr. 291 (2018), 243-262.
- L.R. de Freitas, Multiplicity of solutions for a class of quasilinear equations with exponential critical growth, Nonlinear Anal. 95 (2014), 607-624.
- J.M. do Ó, E. Medeiros, U. Severo, On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^{N}$, J. Differential Equations 246 (2009), 1363-1386.
for multiplicity results, and to the references therein.

The existence theorem we prove extends in several directions Theorem 1.1 of [AFS], Theorem 1.4 of [dFMR], Theorem 1 of [doO], Theorem 1.2 of [FN] and Theorem 1.2 of [YP]. The multiplicity result we present generalizes in several directions Theorem 1.2 of

- L.R. de Freitas, Multiplicity of solutions for a class of quasilinear equations with exponential critical growth, Nonlinear Anal. 95 (2014), 607-624.
which deals with the equation in $\mathbb{R}^{N}$

$$
-\Delta_{N} u+|u|^{N-2} u=\lambda h(x)|u|^{q-2} u+\gamma f(u)
$$

where $f$ does not depend on $x \in \mathbb{R}^{N}$ such as

This fact is due to the variational approach used in [dF], which is strongly based on the study of a critical level, when $\lambda=0$. In order to get this critical
level, the use of the homogeneity of the $N$-Laplace operator is also a crucial requirement. In this paper, the presence of the $(p, N)$ operator in $(\mathcal{E})$ does not allow us to adopt the same approach, even if the variational tools are the same. A crucial point in our argument is, among others, the use of a completely new Brézis and Lieb type lemma for exponential nonlinearities. In the proof of the multiplicity theorem we introduce a suitable tricky step analysis somehow inspired by our previous work

- A. Fiscella, P. P., Degenerate Kirchhoff ( $p, q$ )-fractional problems with critical nonlinearities, submitted for pubblication.
With different approach in the fractional 1-dimensional case we have
- O.H. Miyagaki, P. P., Nonlocal Kirchhoff problems with Trudinger-Moser critical nonlinearity, NoDEA Nonlinear Differential Equations Appl. 26 (2019) art. 27, 26 pp.


## Some open questions

Extension of the results in $\mathbb{R}^{N}$ for

$$
\begin{aligned}
& (\mathcal{E})^{-\Delta_{p} u-\Delta_{N} u+|u|^{p-2} u+|u|^{N-2} u} \\
& =\lambda h(x) u_{+}^{q-1}+\gamma f(x, u)
\end{aligned}
$$

to the vectorial case

$$
(\mathcal{S})\left\{\begin{aligned}
&-\Delta_{p} u-\Delta_{N} u+|u|^{p-2} u+|u|^{N-2} u \\
&=\lambda h(x) u_{+}^{q-1}+F_{u}(x, u, v) \\
&-\Delta_{p} v-\Delta_{N} v+|v|^{p-2} v+|v|^{N-2} v \\
&=\mu h(x) v_{+}^{q-1}+F_{v}(x, u, v)
\end{aligned}\right.
$$

where $1<p<N<\infty, N \geq 2,1<q<N$, $u_{+}=\max \{u, 0\}$, and $h$ is a positive function of class $L^{\theta}\left(\mathbb{R}^{N}\right)$, with $\theta=N /(N-q)$, while $\lambda$ and $\mu$ are positive parameters. The functions $F_{u}, F_{v}$ are partial derivatives of a Carathéodory function $F$, of exponential type.

