Keplerian orbits through the Conley-Zehnder index

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Incontro su Variational methods, with applications to problems in mathematical physics and geometry

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Initial Remarks

- The study of the dynamics of n-point masses interacting according to Newtonian gravitational potential is usually called the **n-body problem**

- From a geometrical point of view a key point consists in trying **to understand the structure of the phase space** looking for the equilibrium points, periodic orbits, invariant tori and in particular their **stability properties**
  The stable and unstable manifolds associated to these objects form a kind of **network of connections** (actually homoclinic, heteroclinic and halfclinic connections), which all together constitute a big part of the essential skeleton of the system

Outline of the talk

- In the first part of the talk we introduce the problem and we briefly discuss the existence of colliding and non-colliding periodic solutions of the n-body problem. (Lagrange relative equilibria and choreographies).

- In the second part of the talk we discuss the Morse index and stability properties of the two-body problem through the Conley-Zehnder index.

n-point interacting systems

We consider $n \geq 2$ point particles with masses $m_1, \ldots, m_n$ and coordinates $q_1(t), \ldots, q_n(t)$ moving in $\mathbb{R}^d$ under the force field induced by a potential function

$$U : \mathbb{X} \to \mathbb{R}$$

where $\mathbb{X} = \mathbb{R}^{nd} \setminus \Delta$ is the configuration space and

$$\Delta := \bigcup_{i,j=1}^{n} \{q \in \mathbb{R}^{nd} | q_i = q_j \text{ for some } i \neq j\}$$

is the collision set (actually an arrangements of hyperplanes).

A class of singular potentials

We focus on the following class of potential functions

- α-homogeneous (gravitational case: $\alpha = 1$):

$$U_\alpha(q) := \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i m_j}{|q_i - q_j|^\alpha}, \qquad \alpha \in (0,2)$$

By the conservation laws of the system, we fix the centre of mass at the origin: $\sum_{i=1}^n m_i x_i = 0$ and we define $I(x) := \|x\|_M^2 = \langle Mx, x\rangle$ called (moment of inertia). We denote by $\mathcal{E}$ the unit sphere in the mass norm (namely the inertia ellipsoid).

Equations of motion

- Newton's Equations:

$$(NE) \qquad M\ddot{q}(t) = \nabla U\big(q(t)\big)$$

where $M := \mathrm{diag}(m_1 I_d, \ldots, m_n I_d)$ is the mass matrix.

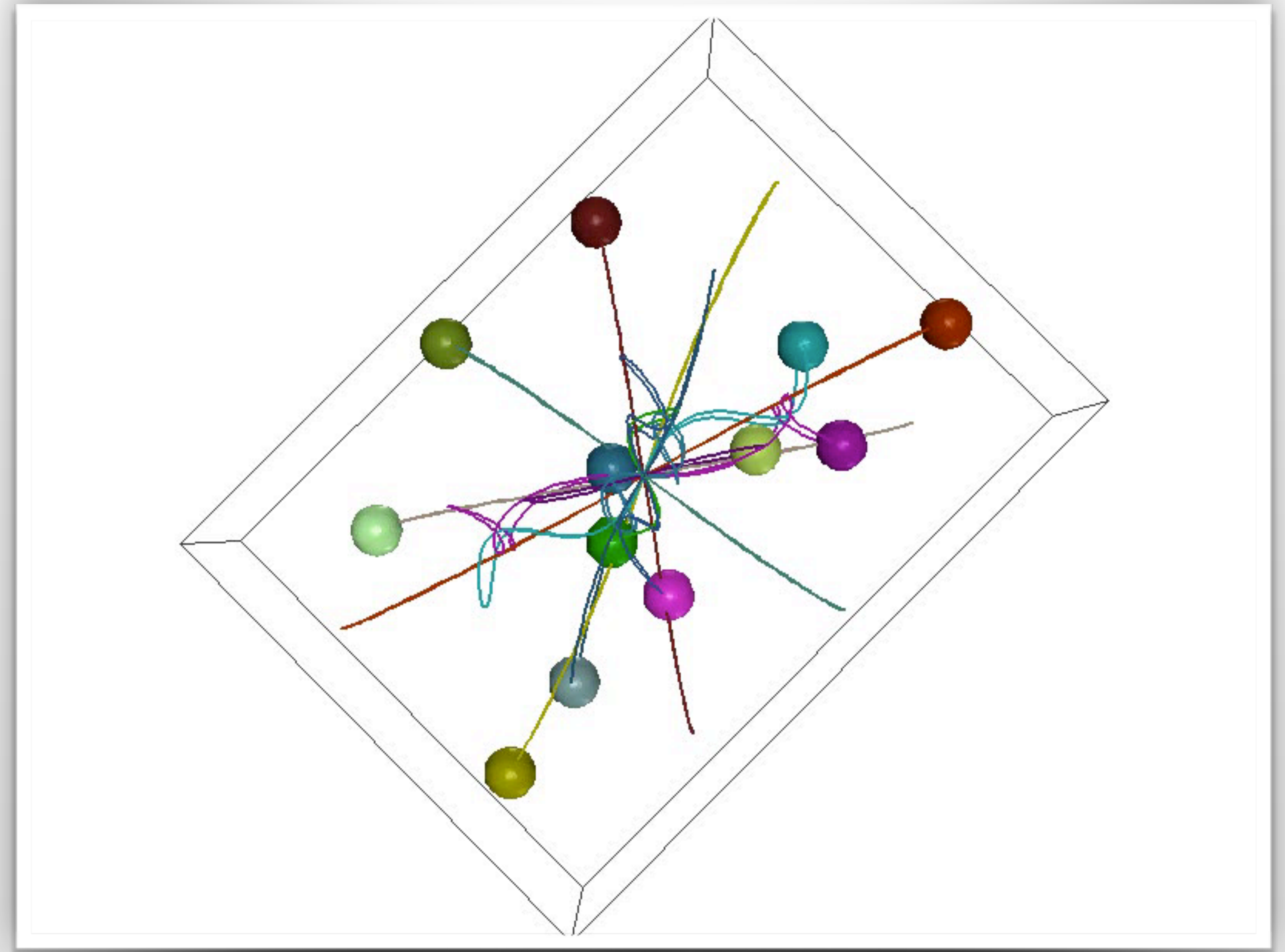
General Remarks The singular set plays a fundamental role in the phase portrait and strongly influence the global orbit structure, being responsible, of the presence of chaotic motions and of motions becoming unbounded in a finite time (Diacu, Devaney, Gerver, Gutzwiller, Mather, Saari, Simò, Xia). Morevore, singularities are intimately linked to the variational structure of periodic trajectories.

Orbits of the (classical) n-body problem

An orbit of the system is a vector valued function $(q_1(t), \ldots, q_n(t))$ of twice differentiable functions which verify the (NE) at each time $t \in (a, b)$. To be meaningful, we have to require $q_i(t) \neq q_j(t)$ for every $t \in (a, b)$. This requirement prevents collisions among the bodies.

- The two body problem is integrable.
- The three body problem cannot be solved and cannot be seen, in its full generality, as a perturbation of a simple integrable problem.

Orbit experiencing a total collapse



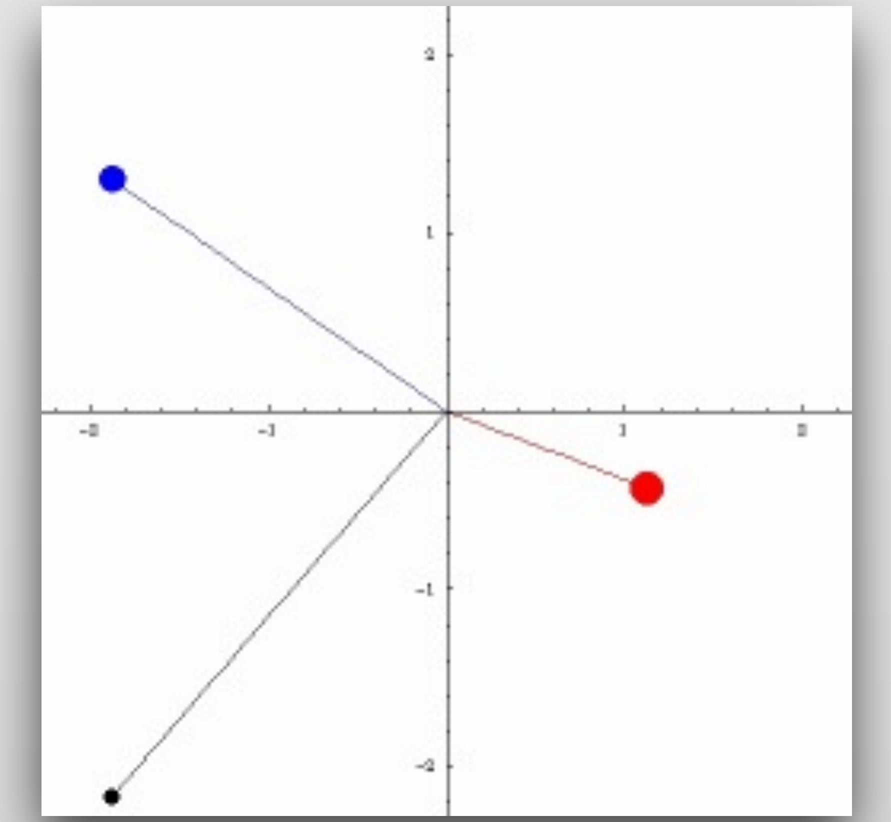
Some related papers adopting a variational approach. Ambrosetti, Bahri, Barutello, Bessi, Chenciner, Chen, Coti Zelati, Desolneux, Ferrario, Gordon, Long, Marchal, Montgomery, Offin, P., Rabinowitz, Riahi, Serra, Tanaka, Terracini,Verzini.

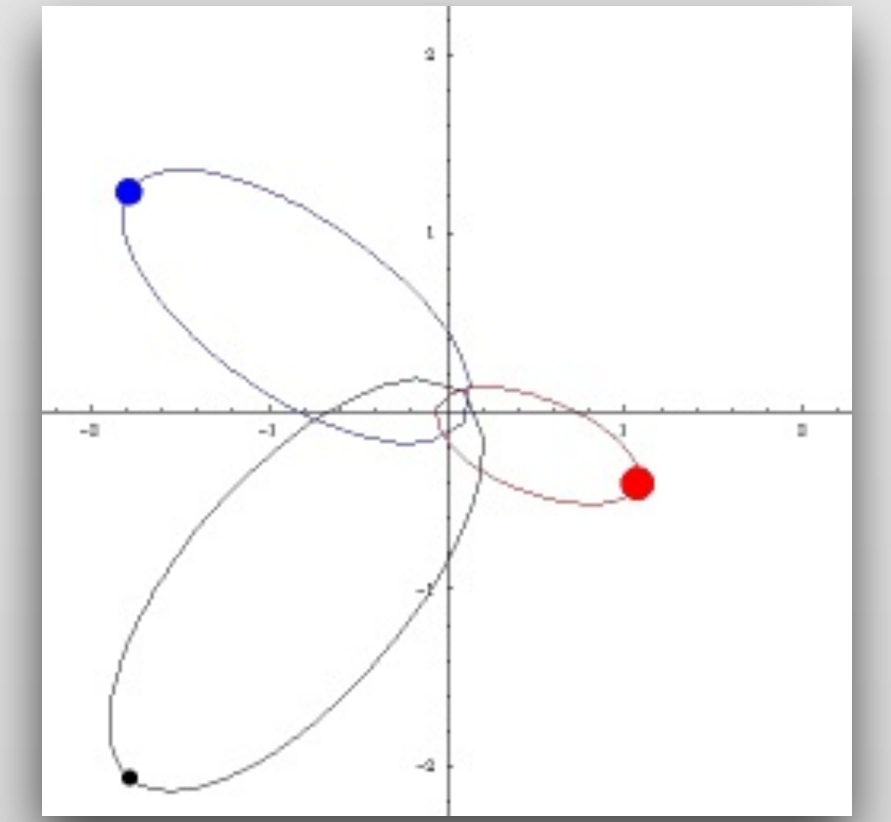
Homographic motions and relative equilibria

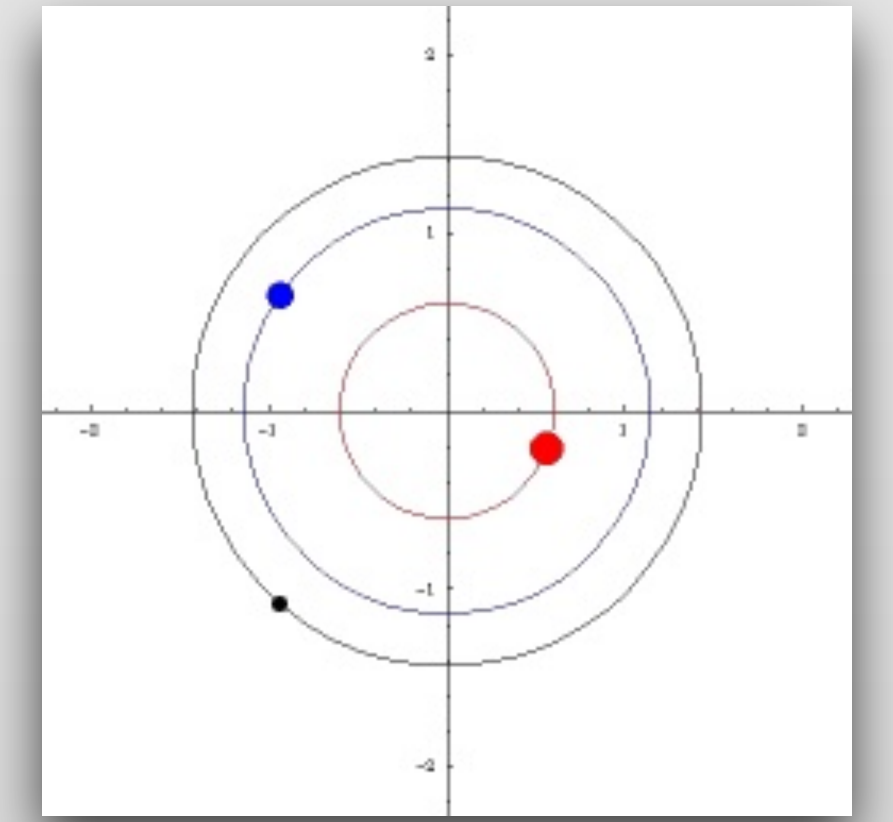
The simplest periodic solutions are associated with central configurations and called homographic orbits.

Main properties

They keep a constant shape (up to rotations and dilations). In such particular motions, each body moves under the effect of one single centre of attraction, located in the barycenter, hence describing an ellipse (or a parabola, or a hyperbola).

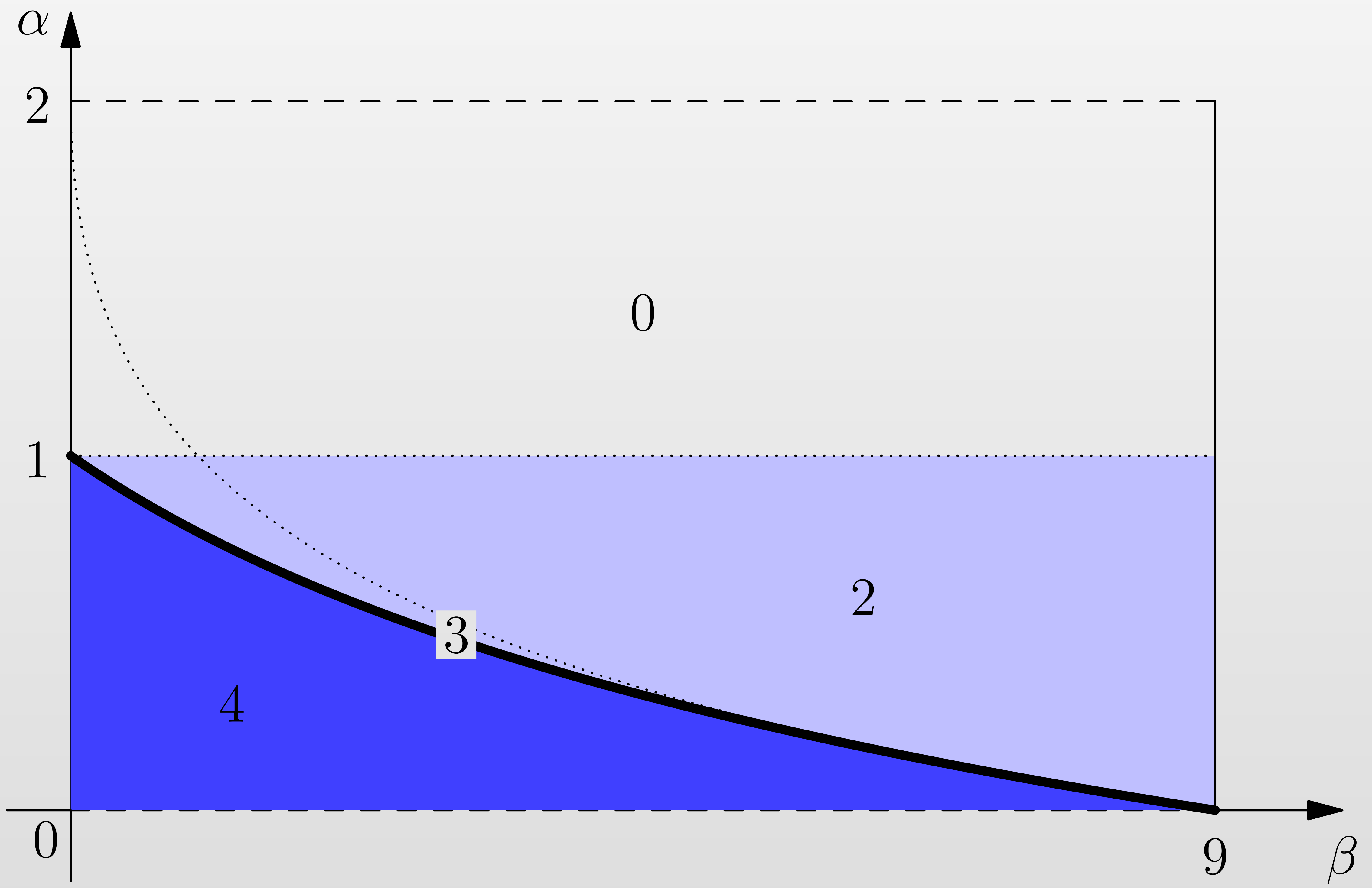






The Morse index of the Lagrange circular orbit

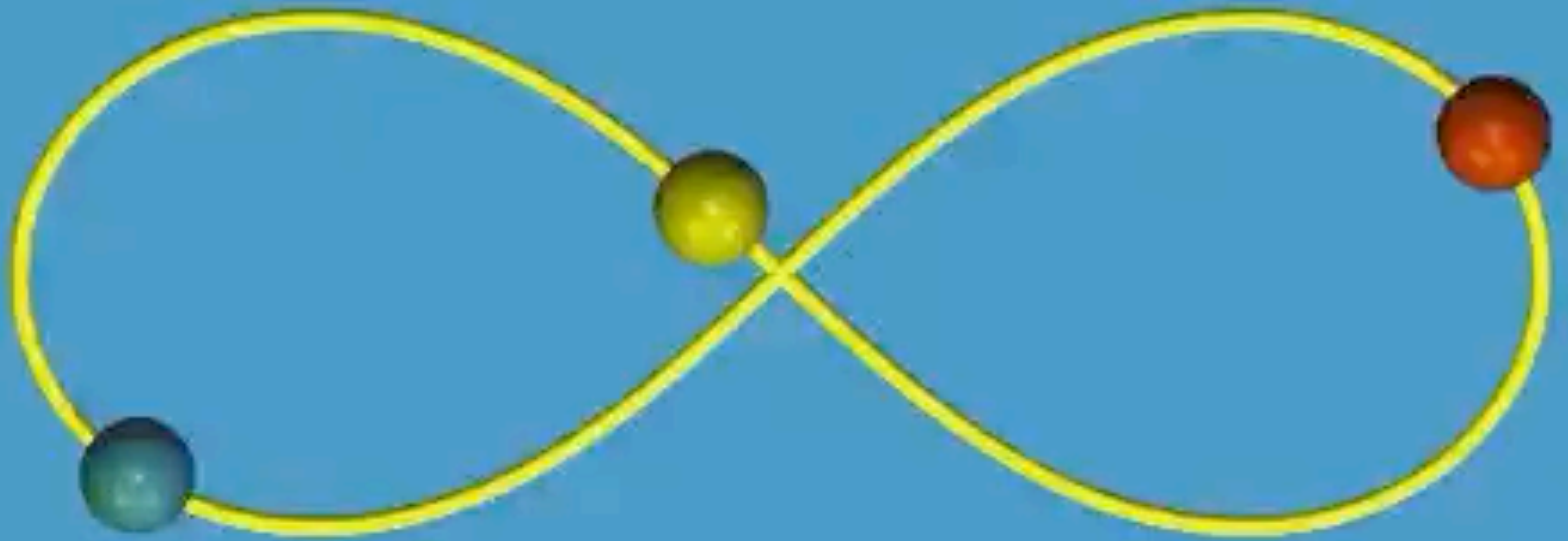
Theorem B The Morse index of $L_{\alpha,\beta}$, as a critical point of the Lagrangian action functional, is



Reference: Barutello, Vivina; Jadanza Riccardo D.; P.A. Morse index and linear stability of the Lagrangian circular orbit in a three-body-type problem via index theory Arch. Ration. Mech. Anal. 219 (2016), no. 1, 387--444.

Figure eight orbit

In recent years, many new periodic orbits orbits have been discovered, using the symmetries (in space and time) and the least action principle. In 2000, two mathematicians Alain Chenciner and Richard Montgomery used the least action principle (Lagrange) with symmetries to find a surprising periodic orbit for the three bodies



Keplerian ellipses in Calculus of Variations

[Gordon, 1977] the infimum of the Lagrangian action on the loop space of $\mathbb{R}^2 \setminus \{0\}$ and having non-vanishing winding number about the origin is realized by the Keplerian orbits (ellipses), including the limiting case of the elliptic collision-ejection orbit which passes through the origin.

The proof is based on the Tonelli Direct Method in Calculus of Variations. The main difficulties are due to the lack of compactness:

- non-compactness due to the unboundedness of the configuration space (cured by minimizing the action on the path connected components of the loop space of $\mathbb{R}^2 \setminus \{0\}$ having winding number ± 1)
- presence of the singularity at the origin (required an ad-hoc asymptotic analysis)

Classification of the Keplerian Ellipses via Conley-Zehnder index

The talk is based on a recent joint paper with Kavle and Offin (preprint available at https://arxiv.org/abs/1908.00075). In this paper we provide

- a new (symplectic) proof of Gordon results through the use of the Conley-Zehnder index
- Gordon's theorem breaks down if the dimension of the configuration space is bigger than 2, contrary to what happens with our approach.

Main Results

Theorem. Let γ be a Keplerian ellipses with prime period T and let γ^k the k-th iteration. Then, we have

$$\iota_{Mor}(\gamma^k) = 2(k-1).$$

In particular

$$\iota_{Mor}(\gamma) = 0.$$

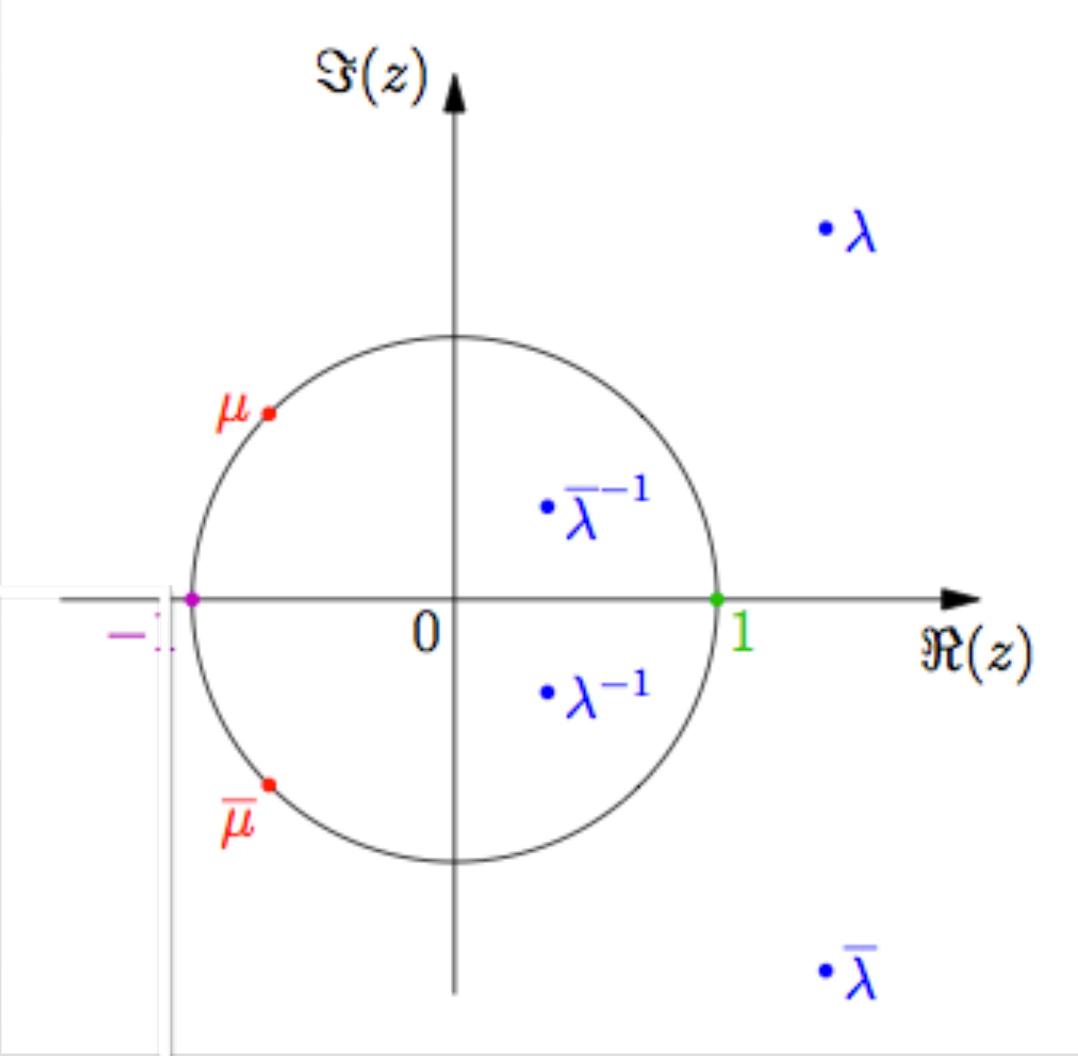
Theorem. *Let γ be a Keplerian ellipse. Then it is elliptic, meaning that all the eigenvalues belongs to the unit circle of the complex plane. Moreover it is spectrally stable and not linearly stable.*

Linear and spectral stability

The study of the stability properties of a Keplerian ellipses corresponds to locate the spectrum of the associated monodromy matrix

Since the monodromy matrix M of a Hamiltonian system is a symplectic matrix, in particular we get that

- $\lambda \in \sigma(M) \Leftarrow \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1} \in \sigma(M)$



A Keplerian ellipses is

- spectrally stable if $\sigma(M) \subset \mathbb{U}$ (unitary circle);
- linearly stable if it is spectrally stable and diagonalizable
- degenerate if $1 \in \sigma(M)$.

Reduction of the two body problem to the 1-center problem

Two point particles in the Euclidean plane having masses $m_1, m_2 \in \mathbb{R}^+$ is mathematically equivalent to the motion of a single body with a reduced mass equal to

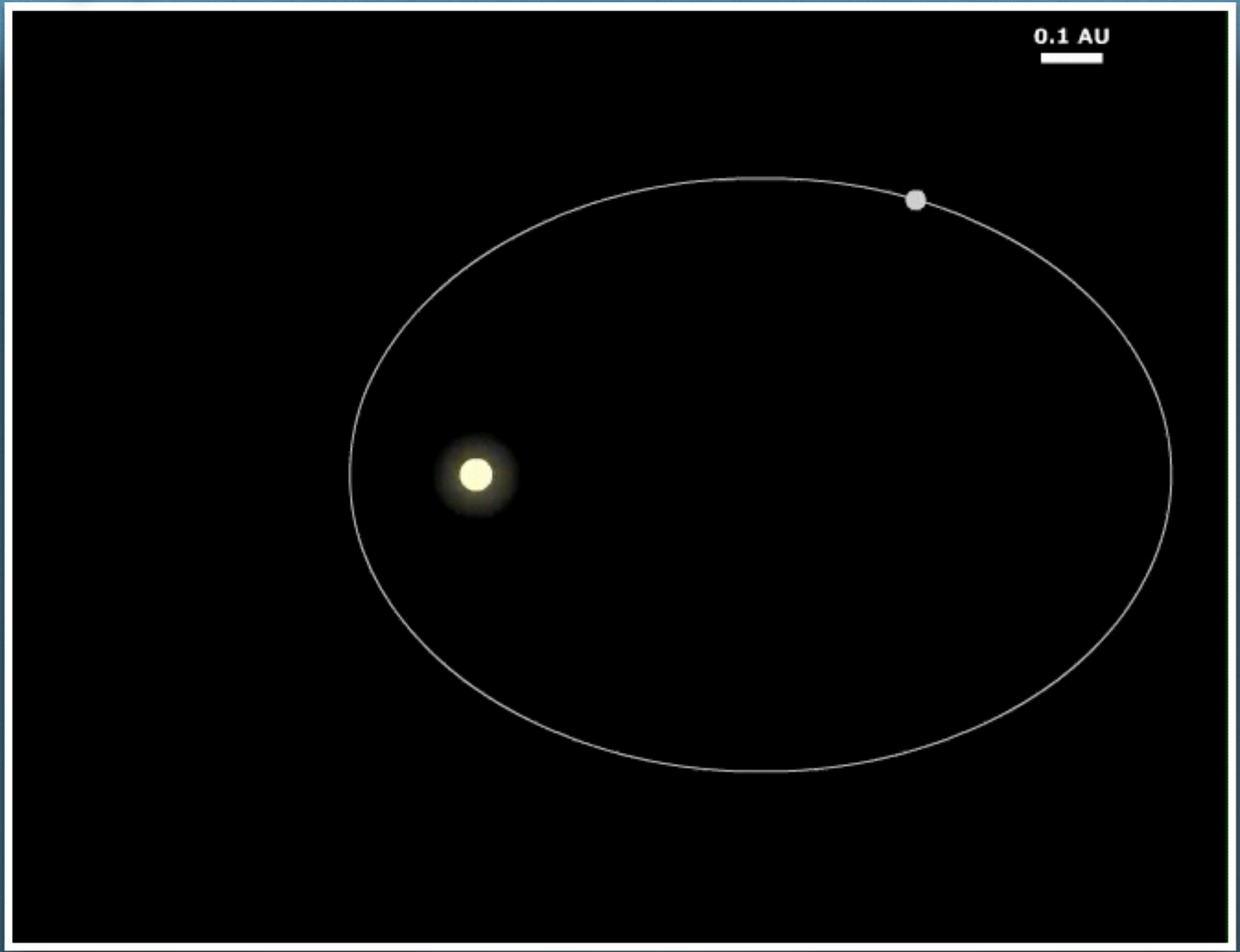
$$\mu := \frac{m_1 m_2}{m_1 + m_2}$$

$\mathbb{X} = \mathbb{R}^2 \setminus \{0\}$. Lagrangian function $L : T\mathbb{X} \to \mathbb{R}$ defined by

$$L(q, v) = K(v) + U(q) \quad \text{where} \quad K(v) = \frac{1}{2}\mu|v|^2, \quad \text{and} \quad U(q) = \frac{m}{|q|}.$$

$m = G m_1 m_2$ and $G \in \mathbb{R}^+$ is the *gravitational constant*.

The two body problem reduces to the classical Kepler problem



Strategy for computing the Morse Index

For calculating the Morse Index

- we compute the Conley-Zehnder index $\iota^{CZ}(\phi_0)$ (a symplectic invariant associated to the fundamental solution ϕ_0 of the linearized Hamiltonian system)
- we use an index theorem that compares these two objects; more precisely

$$\iota_{Mor}(\gamma) = \iota^{CZ}(\phi_0).$$

Remark. Index theorems has a long history that can be traced back to the classical *Sturm oscillation theorem* and the *Morse Index Theorem* in Riemannian geometry.

Polar coordinates in the state space

- Polar coordinates (r, ϑ) on $\mathbb{X}$. The Lagrangian is

$$L(r, \vartheta, \dot{r}, \dot{\vartheta}) = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + U(r).$$

- Euler-Lagrange Equation

$$\begin{cases} \mu\ddot{r} - \mu r\dot{\vartheta}^2 + \dfrac{m}{r^2} = 0 \quad \text{on} \quad [0, T] \\ \dfrac{d}{dt}\left(\mu r^2\dot{\vartheta}\right) = 0. \end{cases}$$

An explicit parametrization

All solutions can be written in terms of the orbital elements; in the particular case of non-zero angular momentum and negative energy, such solutions are ellipses given by given by

$$r(\vartheta) = \frac{r_0}{1 - \varepsilon \cos \vartheta} \quad \text{where } r_0 := \frac{k^2}{\mu m}.$$

- r_0 is called the **semi-latus rectum** and it is related to the eccentricity and the semi-major axis a of the ellipses by $a = r_0/(1-\varepsilon^2))$.

- k is the **angular momentum** and $\varepsilon = \sqrt{1 + \frac{2hk^2}{\mu m^2}}$ (In the circular case $\varepsilon = 0$).

Linearization along a circular orbit

The Hamiltonian function is given by

$$H(r,\vartheta,\dot r,\dot\vartheta) = \frac{1}{2}\mu\left(\dot r^2 + r^2\dot\vartheta^2\right) - U(r) = \frac{1}{2}\left[\frac{p_r^2}{\mu} + \frac{p_\vartheta^2}{r^2\mu}\right] - \frac{m}{r}$$

where $(p_r, p_\vartheta) = (\mu\dot r, \mu r^2\dot\vartheta)$.

Setting $w = (y_r, y_\vartheta, x_r, x_\vartheta)^\mathsf{T}$, then the linearized Hamiltonian system at the circular solution $r(t) = re^{i\omega t}$ is $\dot w = Lw$ where L is the Hamiltonian matrix

$$L = \begin{bmatrix} 0 & \dfrac{2p_\vartheta}{\mu r^3} & \left[\dfrac{2m}{r^3} - \dfrac{3p_\vartheta^2}{\mu r^4}\right] & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{1}{\mu} & 0 & 0 & 0 \\ 0 & \dfrac{1}{r^2\mu} & -\dfrac{2p_\vartheta}{r^3\mu} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A & C & 0 \\ 0 & 0 & 0 & 0 \\ D & 0 & 0 & 0 \\ 0 & B & -A & 0 \end{bmatrix}$$

The infinitesimal phase flow

The matrix L is a time independent Hamiltonian matrix. The fundamental (matrix) solution is given by $\phi_0(t) = e^{Lt}$ where $t \in \left[0, \frac{2\pi}{\omega}\right]$.

Since the determinant of $L - \lambda Id_4$ is $\lambda^2(\lambda^2 + \omega^2)$, there exists a symplectic matrix $P \in \mathrm{Sp}(4)$ such that

$$L = P\left(\begin{bmatrix} 0 & s(r) \\ 0 & 0 \end{bmatrix} \diamond \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\right) P^{-1}.$$

Advantages

- Computation of the Conley-Zehnder index in $\mathrm{Sp}(2)$ instead of $\mathrm{Sp}(4)$
- The Conley-Zehnder index is additive with respect to the symplectic sum $\diamond$

Cylindrical representation of $\mathrm{Sp}(2)$

The symplectic group $\mathrm{Sp}(2)$ captured the attention of I. Gelfand and V. Lidskii first, who in 1958 described a toric representation of it. The $\mathbb{R}^3$-cylindrical coordinate representation of $\mathrm{Sp}(2)$ was introduced by Y. Long in 1991.

Through the polar decomposition, every matrix $M \in \mathrm{Sp}(2)$ can be written as

$$M = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

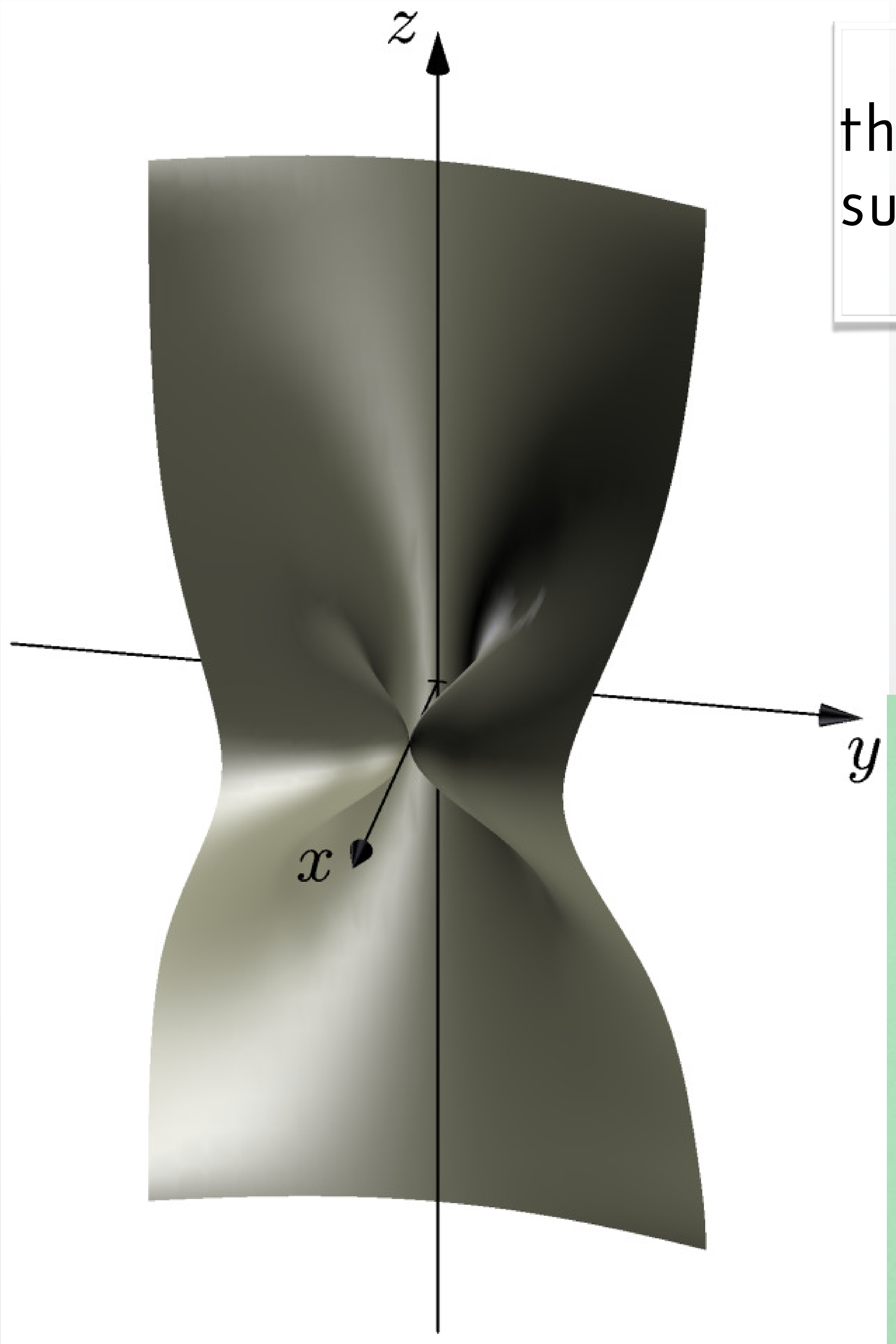
where $(r, \theta, z) \in (0, +\infty) \times [0, 2\pi) \times \mathbb{R}$. Viewing (r, θ, z) as cylindrical coordinates in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ we obtain a smooth global diffeomorphism $\Psi : \mathrm{Sp}(2) \to \mathbb{R}^3 \setminus \{z\text{-axis}\}$. We shall henceforth identify elements in $\mathrm{Sp}(2)$ with their image under Ψ.

Conley-Zehnder index in Sp(2)

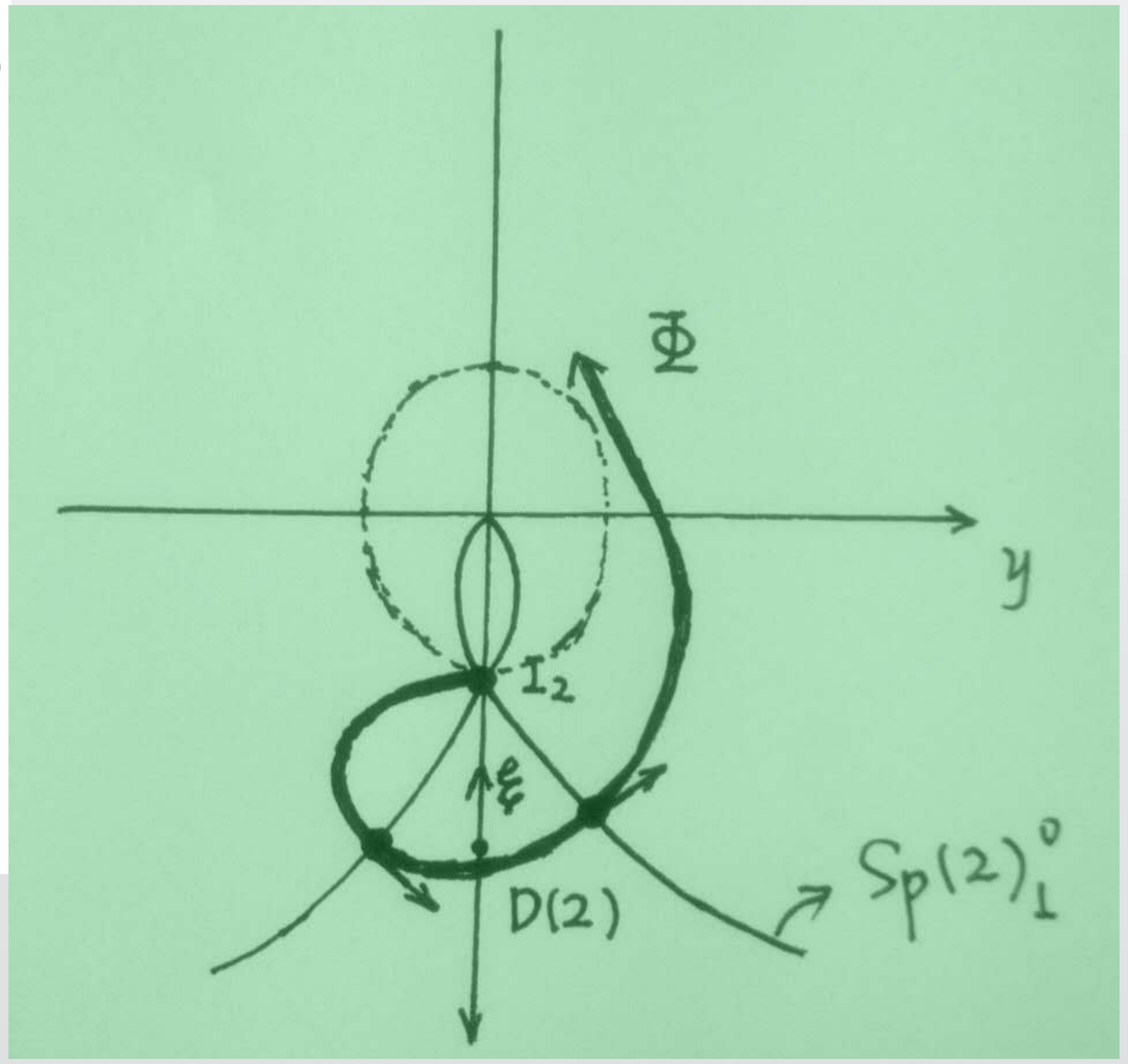
Roughly, the Conley-Zehnder index of a symplectic path in Sp(2) starting from the identity counts the algebraic (signed) intersections of the path with the surface

$$\mathrm{Sp}^0(2) = \{M \in \mathrm{Sp}(2) \mid \det(M - I_2) = 0\}$$

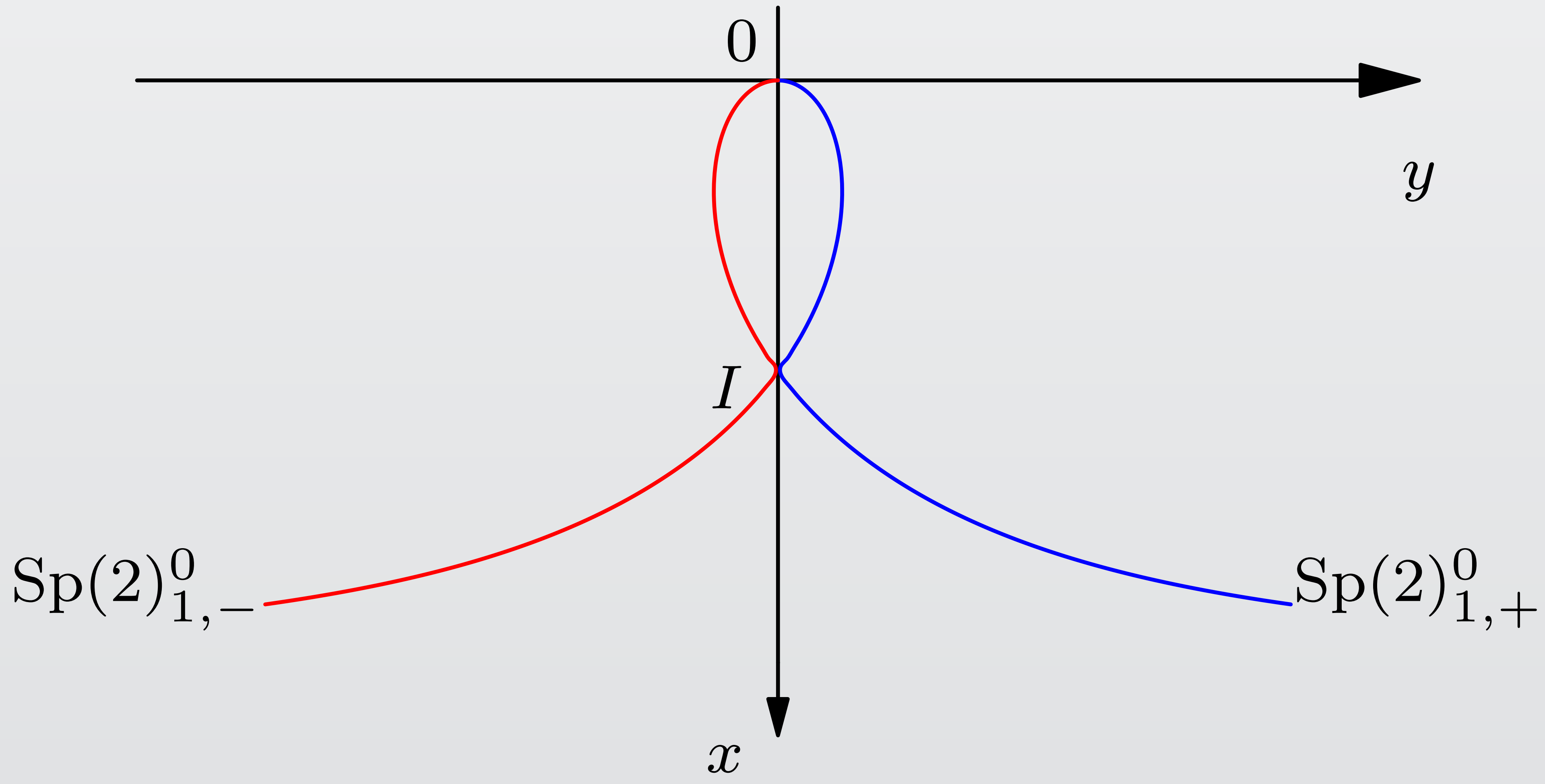
If we imagine projecting the symplectic path into the horizontal plane $\{z = 0\}$ a simple way to think about this index is an algebraic count of intersection with the curve depicted in Figure of a continuous path (which is the projection of the original one) on the $\widehat{xOy}$-plane.



Arnol'd-Maslov cycle in Sp(2)



A path having CZ-index =2



Intersection between the Arnol'd-Maslov cycle and the plane z=0

Homotopy arguments and symplectic phase flow invariant decomposition

- The linearized Hamiltonian vector field is constant
- Up to a symplectic change of coordinates, we get that $L \sim K_1 \diamond K_2$ where

$$K_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

- We let $\phi_0 : [0, T] \to \mathrm{Sp}(4)$ the fundamental solution of the Hamiltonian system $\dot{w} = Lw$
- For $i = 1, 2$, we let $\phi_i : [0, T] \to \mathrm{Sp}(2)$ be the fundamental solutions of the Hamiltonian systems $\dot{w} = K_i w$

Fundamental solutions and monodromy

By a direct integration, we get that

- $\phi_1(t) = \begin{bmatrix} 1 & s(r)t \\ 0 & 1 \end{bmatrix} \qquad t \in [0, 2\pi/\omega]$
- $\phi_2(t) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \qquad t \in [0, 2\pi/\omega]$

where $s(r) = 3/r^2 > 0$

The monodromy matrix is

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \diamond \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Stability of circular motions

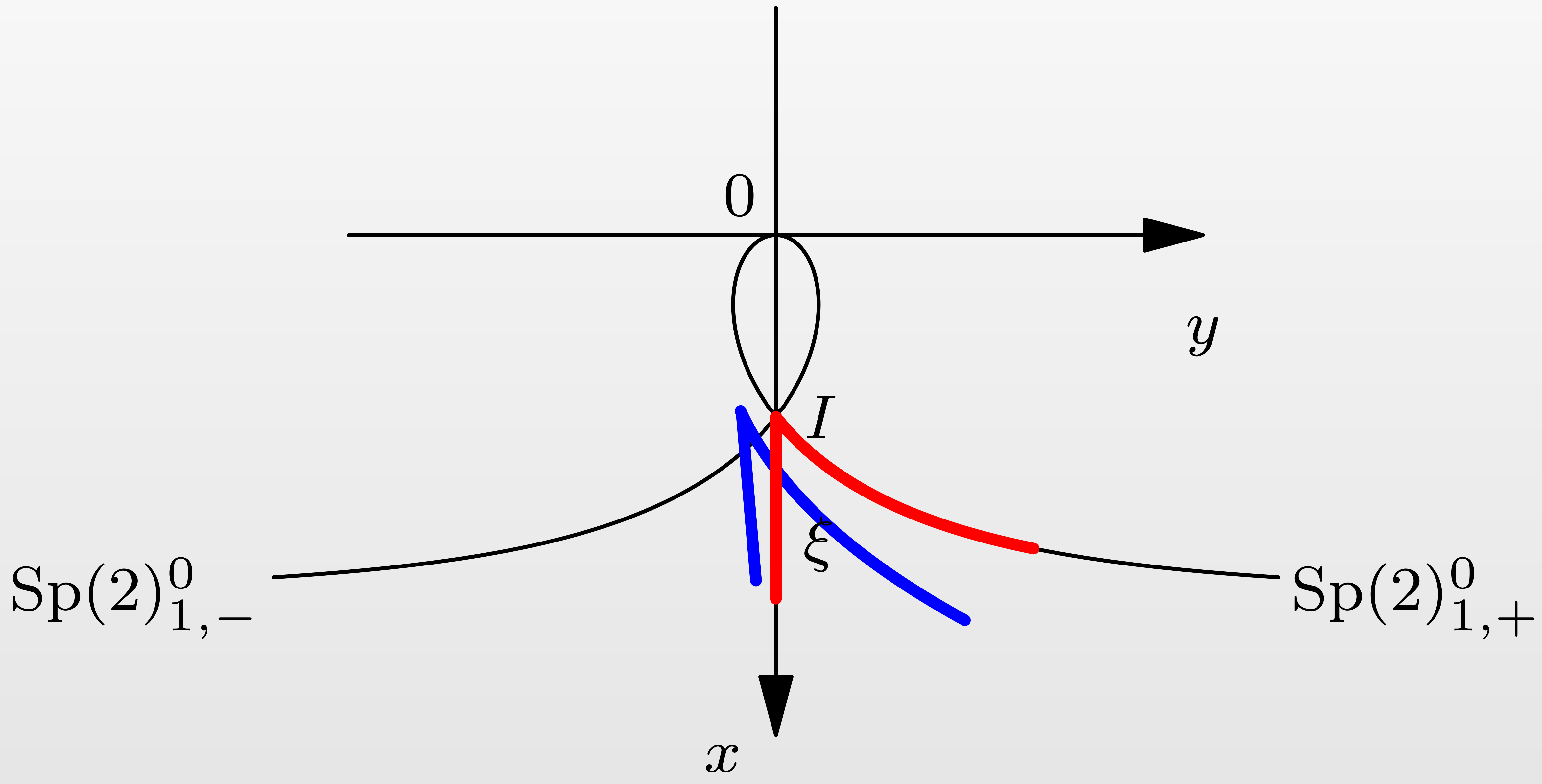
Thus, we get that

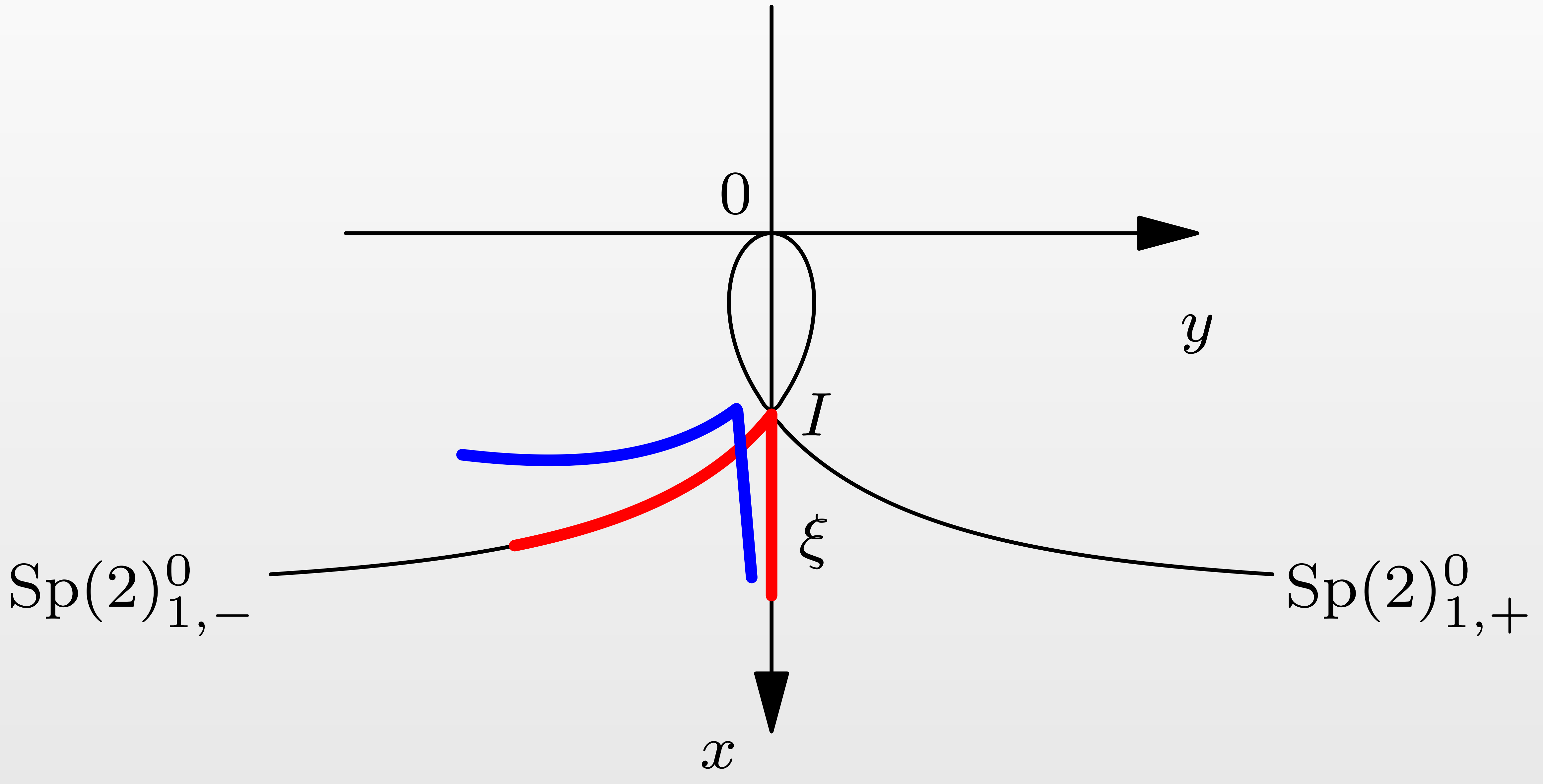
- $\sigma(M) = \{1\} \in \mathbb{U}$ and the algebraic multiplicity of its (unique) Floquet multiplier is 4.
- M is not diagonalizable (having a non-trivial Jordan block); thus in particular γ is spectrally and not linearly stable. We also observe that its nullity is $3 = \dim\ker(M - I)$.

Remark. We observe that the 2×2

- Jordan block relative to the eigenvalue 1 is the symplectic normal form corresponding to the energy conservation law
- identity matrix is the symplectic normal form corresponding to the angular momentum conservation law

Conley-Zehnder index of the path ϕ_1





Vanishing CZ-index for the path on the left
CZ-index= -1 for the path on the right

The path in red and its deformation in blue. On the left corresponding to a negative upper right entry (respectively right corresponding to a positive upper right entry), the first path starts at the matrix $D(2)$ direct towards the identity, then goes downwards left. The second path (corresponding to the ϵ-perturbed one) follows the same trajectory, just rotated clockwise by an angle ϵ. The ι^{CZ}-index of both paths on the left is 0. The ι^{CZ}-index of both paths on the right is -1.

Conley-Zehnder index of circular motion

- $\iota^{CZ}(\phi_0(t), t \in [0,T]) = \iota^{CZ}(\phi_1(t), t \in [0,T]) + \iota^{CZ}(\phi_2(t), t \in [0,T]$
- $\iota^{CZ}(\phi_2(t), t \in [0,T] = 2\left\lfloor \frac{\omega T}{2\pi} \right\rfloor - 1$ if $T \in \frac{2\pi}{\omega}\mathbb{Z}$
- $\iota^{CZ}(\phi_1(t), t \in [0,T]) = -1$ (independent on T)

Summing up we get

$$\iota^{CZ}(\phi_0(t), t \in [0,T]) = 2\left\lfloor \frac{\omega T}{2\pi} \right\rfloor - 2 \text{ for } T \in \frac{2\pi}{\omega}\mathbb{Z}$$

Conley-Zehnder index for a Keplerian ellipses

Given a periodic Kepler orbit $z_{h_0}(t)$ having energy h_0, there exists $\delta > 0$ and a smooth one-parameter family of periodic orbits $z_h(t)$ with $H(z_h(t) = h$ for all $h \in (h_0 - \delta, h_0 + \delta)$.
Let T_h be the period of z_h, ϕ_h be the corresponding fundamental solution, $M_h = \phi_h(T_h)$ be the monodromy matrix and $M := M_{h_0}$

M is symplectically conjugated to $N_1(1, a) \diamond Id_2$ where $a = 0, \pm 1$. In particular

- if $\frac{dT_h}{dh}|_{h=h_0} > 0$, then $a = 1$

Since $T_h = (2^{-1/2}\pi)(-h)^{-3/2}$, we end-up precisely as in the circular case.

Remark. Analogous result in a different context was used by Long and Ekeland in the study of closed characteristics.

New perspectives and working progress

- [Working progress with D. Offin] Compute the Morse index and the stability properties for the 3D Kepler problem
- [Working progress with S. Terracini] How the presence of total/partial collision can contribute to the Morse index
- [Working progress with A. Abbondandolo] Can holomorphic curve techniques be applied to the three-body problem? Try to develop a suitable Floer-Rabinowitz theory!
- How can the regularization of collisions affect the Morse index? [Try to establish a precise relation between the Maslov index of a colliding trajectory and its regularized (in the sense of Levi-Civita, Moser, etc.)]


THANK YOU
GRACIAS
ARIGATO
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TINGKI
YAQHANYELAY
TASHAKKUR ATU
CHALTU
NUHUN
SPASSIBO
SNACHALHUYA
SUKSAMA
EKHMET
WABEEJA
MAITEKA
HUI
YUSPAGARATAM
MEHRBANI
GRAZIE
MAAKE
KOMAPSUMNIDA
SANCO
MERASTAWHY
GAEJTHO
GOZAIMASHITA
EFCHARISTO
AGUYJE
FAKAAUE
TAVTAPUCH
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