On the stability of the Gauss Mean Value formula for harmonic functions

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A rigidity property of the Euclidean balls

- D open $\subseteq \mathbb{R}^n$ $(n \geq 3)$, $|D| < \infty$, $x_0 \in D$

$$u(x_0) = \fint_D u\,dx := \frac{1}{|D|}\int_D u\,dx \qquad \forall u \in H^1(D)$$

$$\Updownarrow$$

D = Euclidean ball centered at x_0

- " $\Uparrow$ " = Gauss Mean Value Theorem
- " $\Downarrow$ " = Kuran's Theorem (1972)

A stability question

- D open $\subseteq \mathbb{R}^n$, $|D| < \infty$, $x_0 \in D$.

If

$u(x_0)$ is close to $\fint_D u$ $\quad \forall u \in H^1(D)$

is it true that

D is close to a Euclidean ball centered at x_0 ?

Gauss gap

- To put the previous question in a precise form we have introduced the Gauss Mean Value gap

- D open $\subseteq \mathbb{R}^n$, $|D| < \infty$, $x_0 \in D$.

$G(D, x_0)$ = Gauss gap of D w.r.t. x_0

$$= \sup_{\substack{u \in H^1(D) \\ u \neq 0}} \frac{\left| u(x_0) - \fint_D u \right|}{\|u\|_{\mathcal{H}^1(D)}}$$

- $\|u\|_{\mathcal{H}^1(D)} = \int_D |u|$

Basic properties of G

- $G(D, x_0) \leq 1 + \dfrac{|D|}{|B(x_0, r)|} < \infty \quad \forall \; B(x_0, r) \subseteq D$

- G is translations and dilations invariant

- D = Euclidean ball centered at x_0 $\Longrightarrow$ $G(D, x_0) = 0$

  ↑ Gauss

  Kuran

  $\Downarrow$

  D = Euclidean ball centered at x_0

Agreements

(A1) $u(x_0)$ is close to $\int_D u$ for every $u \in H^1(D)$ iff

$G(D, x_0)$ is small

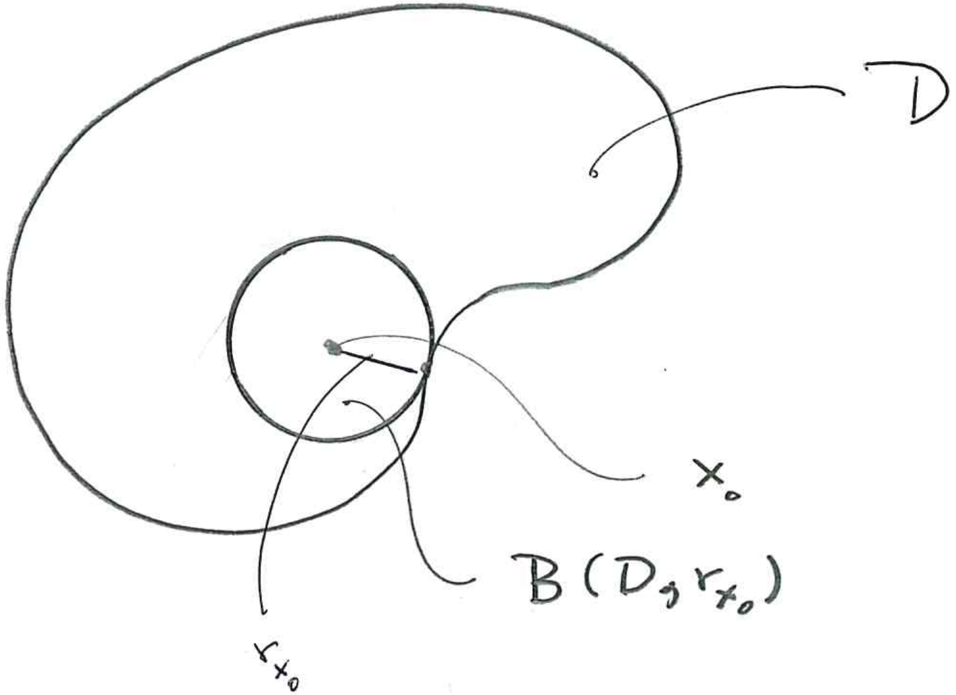
(A2) D is close to a Euclidean ball centered at $x_0 \in D$ iff the ball gap of D w.r.t. x_0, i.e.

$$b(D, x_0) := \frac{|D \setminus B(x_0, r_{x_0})|}{|D|}, \quad r_{x_0} = \mathrm{dist}(x_0, \partial D)$$

is small

NOTE $b(D, x_0) = 0 \Rightarrow D = B(x_0, r_{x_0})$

The ball gap w.r.to x_0



$$b(D, x_0) = \frac{|D \setminus B(x_0, r_{x_0})|}{|D|}$$

The formal question

With (A1) and (A2) at hands

our question can be formulated as follows :

If $G(D, x_0) \longrightarrow 0$

can we say that

$$b(D, x_0) \longrightarrow 0 \quad ?$$

The answer is YES !

Our stability result

Theorem 1. D open $\subseteq \mathbb{R}^n$, $x_0 \in D$ $\Longrightarrow$

(SI) $$G(D, x_0) \geq c(n)\, b(D, x_0)$$

$c(n) > 0$ only depends on n.

NOTE (SI) $\Longrightarrow$ Kuran's Theorem

Sharpness of (SI)

- B = Euclidean ball centered at x_0
- $\mathcal{E}$ = family of the ellipsoids $E \supseteq B$, E centered at x_0, $|E| < 2|B|$

Then

$$c(n)\, b(E, x_0) \leq G(E, x_0) \leq a(n)\, b(E, x_0)$$

$\uparrow$ (SI) $\qquad\qquad$ $\uparrow$ (*)

$a(n) > 0$ only depends on n.

$C^{1,\alpha}$- continuity of G

The previous inequality (*) comes from

Theorem 2: Let

$$D = \{ x \in B(0,2) : d(x) < 1 \}, \quad 0 \in D,$$

where $d \in C^{1,\alpha}(B(0,2))$, $0 < \alpha < 1$. Suppose

$$\partial D = \{ d = 1 \} \quad \text{and} \quad B(0, \tfrac{1}{2}) \subseteq D \subseteq B(0,2).$$

Let $d_e : \mathbb{R}^n \to \mathbb{R}$, $d_e(x) = |x|^2$. Then

$$(G) \qquad G(D,0) \leq c \, \| d - d_e \|_{C^{1,\alpha}(B(0,2))}$$

$c > 0$ only depends on n and on the $C^{1,\alpha}$- norm of d.

Optimality of $C^{1,\alpha}$ in (G)

$C^{1,\alpha}$ in (G) cannot be replaced with C^{β}, $\beta \in]0,1[$,

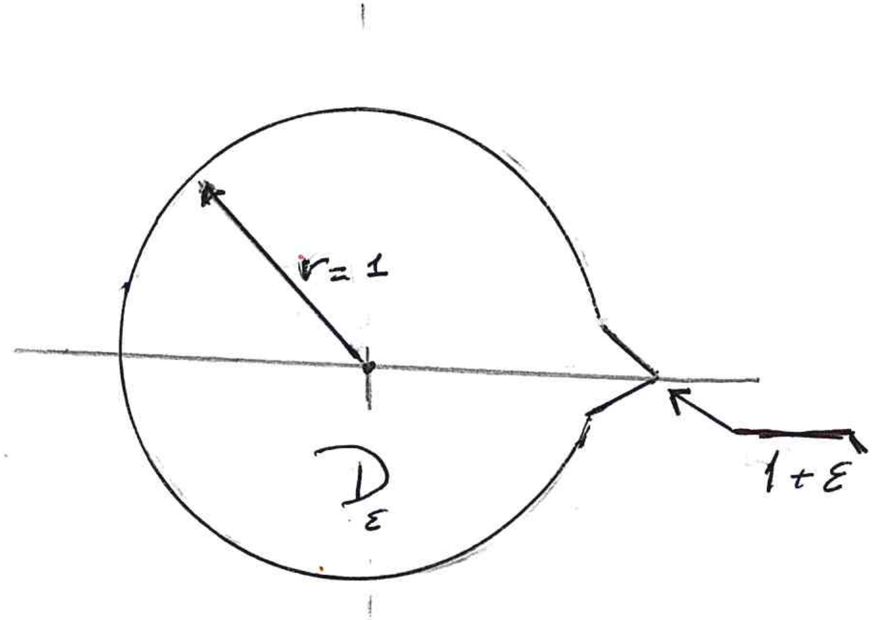
or even with $W^{1,p}$, $1 \le p < \infty$.

Actually we construct a family of equi-Lipschitz domains $(D_\varepsilon)_{\varepsilon > 0}$,

$$D_\varepsilon = \{ x : d_\varepsilon(x) < 1 \}, \quad D_\varepsilon \ni 0,$$

such that

$$\lim_{\varepsilon \to 0} \frac{G(D,0)}{\| d_\varepsilon - d_e \|_{W^{1,p}}} = \infty \quad \forall p \in [1, \infty[.$$


r = 1
D
ε
1 + ε

To obtain an upper extimate

of the Gap between

$u(x_0)$ and the average $\int_D u \quad (D \ni x_0)$

in terms of $W^{1,p}$- norms

it seems more appropriate to use a

Gauss gap in L^p , $1 \leq p \leq \infty$

The G^p-gap

Let D open $\subseteq \mathbb{R}^n$, $|D| < \infty$, and let $x_0 \in D$.
If $1 \le p \le \infty$, define

$$G^p(D, x_0) := \sup_{0 \neq u \in H^p(D)} \frac{|u(x_0) - \fint_D u|}{\|u\|_{L^p(D)}}$$

- $H^p(D) = \{ u \text{ harmonic in } D : \int_D |u|^p < \infty \}$
- $\|u\|_{L^p(D)} := \left(\fint_D |u|^p \right)^{1/p}$ if $1 \le p < \infty$
- $\|u\|_{L^\infty(D)} := \sup_D |u|$

Basic properties of G^p

- $G^1 = G$

- $1 \leq p \leq q \leq \infty \Rightarrow$

$$G^1(D, x_0) \geq G^p(D, x_0) \geq G^q(D, x_0) \geq G^\infty(D, x_0)$$

- G^p is translations and dilations invariant

- $B =$ ball centered at $x_0 \Rightarrow G^p(B, x_0) = 0$

- If $1 < p \leq \infty$ there exist $u \in H^p(D)$:

$$G^p(D, x_0) = \frac{|u(x_0) - f_D u|}{\|u\|_{t^p(D)}}$$

G^p- stability estimate

Theorem 3.- $1 \leq p < \frac{n}{n-1} \Longrightarrow$

$$G^p(D, x_0) \geq c(n, p)\, b(D, x_0).$$

Corollary 4.- $G^p(D, x_0) = 0$ for some $p \in [1, \frac{n}{n-1}[$

$$\Downarrow$$

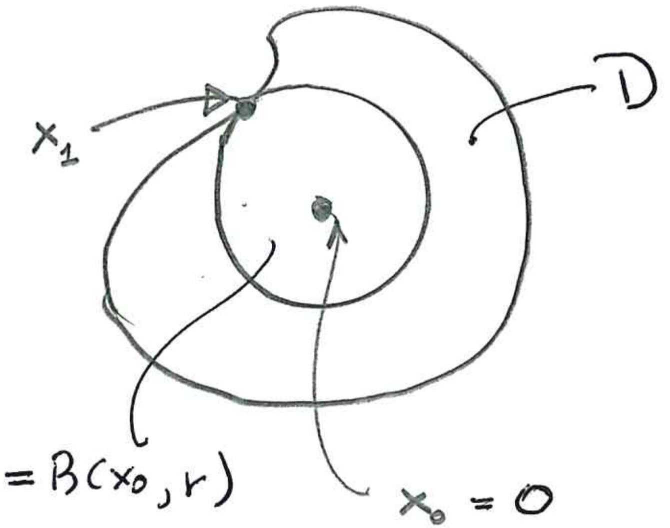
D = Euclidean ball centered at x_0.

Equivalently: $u(x_0) = \fint_D u \quad \forall u \in H^p(D) \Longrightarrow$

D = Euclidean ball centered at x_0.

Proof of Theorem 3 (Main ideas)

- We may assume $x_0 = 0$ (G^p is translations invariant)
- Let $x_1 \in \partial D$ s.t. $|x_1| = |x_1 - x_0| = \operatorname{dist}(x_0, \partial D) =: r$



- Consider the Kuran's function

$$h(x) = 1 + r^{n-2} \frac{|x|^2 - r^2}{|x - x_1|^n}, \qquad x \neq x_1$$

- $h \in \mathcal{H}(\mathbb{R}^n \setminus \{x_1\})$

continuation

- $|h(x)| \leq \dfrac{c}{|x-x_1|^{n-1}} \Rightarrow h \in H^p(D)$ if $1 \leq p < \dfrac{n}{n-1}$

- $h(0) = 0$, $h(x) \geq 1 \quad \forall x \in D \setminus B$

- $G^p(D, x_0) \geq \dfrac{|h(0) - f_D u|}{\|u\|_{L^p(D)}} = c(n,p) \left| f_D h \right|$

$$= c(n,p) \frac{1}{|D|} \left| \int_{D \setminus B} h + \underbrace{\int_B h}_{=0} \right|$$

$$= \frac{c(n,p)}{|D|} \int_{D \setminus B} h \geq c(n,p) \frac{|D \setminus B|}{|D|}$$

$$= c(n,p)\, b(D, x_0)$$

#

$W^{1,p}$- estimate of G^p for $C^{1,\alpha}$- domains

Theorem 5. Let

$$D = \{ x \in B(0,2) \; ; \; d(x) < 1 \}$$

where $d \in C^{1,\alpha}(B(0,2))$, $0 < \alpha < 1$; $d(0) = 0$

Suppose

$$\partial D = \{ d = 1 \} \text{ and } B(0, \tfrac{1}{2}) \subseteq D \subseteq B(0,2)$$

Let $d_e : \mathbb{R}^n \to \mathbb{R}$, $d_e(x) = |x|^2$

If $1 < p < \frac{n}{n-1}$ then

$$G^p(D,0) \leq c \, \| d - d_e \|_{W^{1,q}(D)} \; , \quad \frac{1}{p} + \frac{1}{q} = 1 .$$

$c > 0$ depends on the $C^{1,\alpha}$ norm of d.

Proof (Main ideas)

(I) A density with the mean value property

- $G_D = G_D(0, \cdot)$ = Green functions of D with pole at 0
- $\varphi :]0, \infty[\to]0, \infty[$ s.t. $\int_0^\infty \varphi = 1$
- $W_D := \varphi(G_D) \, |\nabla G_D|^2$

Then

$$u(0) = \int_D W_D(x) u(x) \, dx \quad \forall u \in H^p(D).$$

NOTE. φ is such that $W_D = \dfrac{1}{|B|}$ if $D = B$ = unit ball

continuation

(II) $u \in H^p(D) \Longrightarrow$

$$\left| u(0) - \fint_D u \right| = \left| \int_D \left(\Psi_D - \frac{1}{|D|} \right) u \, dx \right|$$

$$\leq \| u \|_{L^p(D)} \left\| \Psi_D - \frac{1}{|D|} \right\|_{L^q(D)}$$

Then

$$G^p(D, 0) \leq c(n, p) \left\| \Psi_D - \frac{1}{|D|} \right\|_{L^q(D)}$$

continuation

(III)

$$\| W_D - \frac{1}{|D|} \|_{L^q} \leq$$

$$\| W_D - \frac{1}{|B|} \|_{L^q} + \| \frac{1}{|B|} - \frac{1}{|D|} \|_{L^q}$$

$\leftarrow q > n$

$$\leq c \, \| h \|_{W^{1,q}} \qquad\qquad \leq c \, \| d_\varepsilon - d_e \|_{W^{1,q}}$$

$$\begin{cases} \Delta h = 0 & \text{in } D \\ h = \phi(d_e - d_\varepsilon) & \text{on } \partial D \end{cases}, \quad \phi \in C^{1,\alpha}(B(0,2))$$

(IV) D is a <u>Reifenberg-flat</u> domain

$$\Downarrow$$

$$\| h \|_{W^{1,q}} \leq c \| d - d_e \|_{W^{1,q}}.$$

Conclusion :

$$G^{p}(D, 0) \leq c \| d - d_e \|_{W^{1,q}(D)}$$

if

$$1 < p < \frac{n}{n-1}, \quad \frac{1}{q} + \frac{1}{p} = 1 \quad (\text{hence } q > n)$$

A rigidity result in terms of G^p for $\frac{n}{n-1} \le p \le \infty$.

Theorem 6 Let D open $\subseteq \mathbb{R}^n$ and $x_0 \in D$.

Assume $|D| < \infty$ and $D = \mathrm{int}(\overline{D})$

If $G^p(D, x_0) = 0$ for some $p \in [\frac{n}{n-1}, \infty]$,

then

D = Euclidean ball centered at x_0

NOTE See Corollary 4 for $1 \le p < \frac{n}{n-1}$

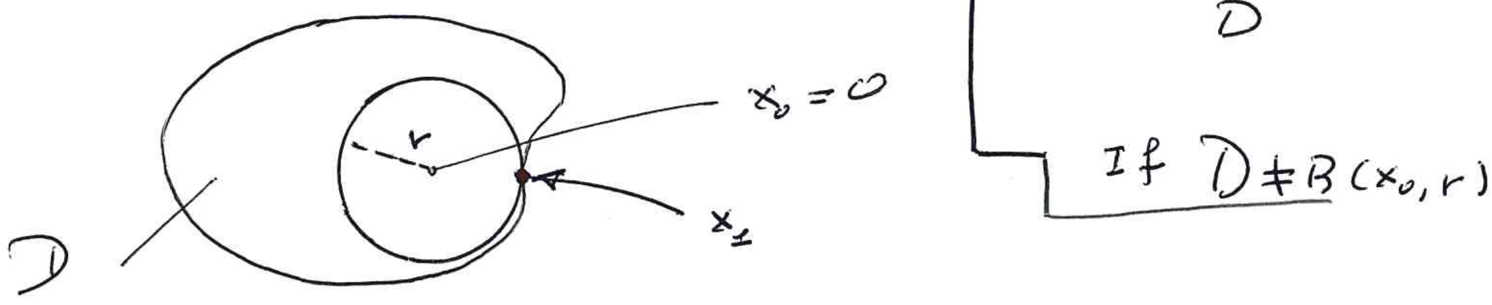
<u>Proof</u> (sketch) For $\alpha \notin \overline{D}$ define

$$h_\alpha(x) := 1 + |\alpha|^{n-2} \frac{|x|^2 - |\alpha|^2}{|x-\alpha|^{n-2}}, \quad x \neq \alpha .$$

- $h_\alpha \in \mathcal{H}^p(D)$ for $1 \leq p \leq \infty$
- Let $x_1 \in \partial D$: $r := |x_1| = \mathrm{dist}(0, \partial D)$ (assume $x_0 = 0$!)

Then h_{x_1} is the Kuran's function and

$$h_\alpha(0) - \fint_D h_\alpha \longrightarrow h_{x_1}(0) - \fint_D h_{x_1} \neq 0 \quad \text{as } \underset{\overline{D}^c}{\alpha \to x_1}$$



continuation

• Then : $G^p(D, x_0) \geq$

$$\frac{|h_\alpha(x_0) - f_D h_\alpha|}{\|h_\alpha\|_{L^p(D)}} > 0 \quad \text{if } \alpha \notin \bar{D}, \ \alpha \text{ \underline{close} to } x_1$$

↑

if $D \neq B(x_0, r)$

Hence

$$G^p(D, x_0) = 0 \Longrightarrow D = B(x_0, r)$$

The Γ-gap : motivation

- $\Gamma(x) = c_n |x|^{2-n}$: fundamental solution of Δ, $n \geq 3$.
- Consider a ball $B = B(x_0, r)$
- The function $B \ni y \longmapsto \Gamma(y-x)$, $x \notin B$

  is harmonic and in L^1, in B.

Then (Gauss Theorem):

$$\Gamma(x_0 - x) = c \int_B \Gamma(y-x)\,dy \quad \forall x \notin B \ ; \quad c = \frac{1}{|B|}$$

continuation

The previous identity can be red as follows:

The Newtonial potential of a ball centered at x_0 is proportional, outside the ball, to the Newtonian potential of a mass concentrated at $\{x_0\}$

This is another rigidity property of the Euclidean ball

Γ- rigidity of the balls

Theorem. Let $D \subseteq \mathbb{R}^n$, open, $|D| < \infty$ and $x_0 \in D$.

Suppose

$$\Gamma(x_0 - x) = c \int_D \Gamma(y-x)\, dy \qquad \forall x \notin D$$

for a suitable $c > 0$.

Then $c = \dfrac{1}{|D|}$ and D = Euclidean ball centered at x_0

- Aharonov-Shiffer-Zalcman (1981) if D is bounded and $D = \mathrm{int}(\overline{D})$

Cupini-Lanc. (2019) if $|D| < \infty$.

The Γ- gap

- D open $\subseteq \mathbb{R}^n$, $|D| < \infty$, $x_0 \in D$.

- $\Gamma(D, x_0) = \Gamma$- gap of D w.r.to x_0

$$:= \sup_{x \notin D} \left| \fint_D \frac{\Gamma(y-x)}{\Gamma(x_0-x)}\, dy - 1 \right|$$

$$= \sup_{x \notin D} \left| \fint_D \left(\frac{|x_0 - x|}{|y-x|} \right)^{N-2} dy - 1 \right|$$

Remarks

- $\Gamma(D, x_0)$ is translations and scale invariant

- $\Gamma(D, x_0) = 0 \iff$

$$\Gamma(x_0 - x) = \fint_D \Gamma(y - x)\, dy \quad \forall x \notin D \iff$$

(Aharonov-Shiffer-Zalcman Th.):

D = Euclidean ball centered at x_0

- OPEN PROBLEM. Stability in terms of Γ-gap

Final Remarks

other rigidity properties of the Euclidean balls

stability open problems

In wath follows:

D bounded open $\subseteq \mathbb{R}^n$, $\partial D \in C^1$, $x_0 \in D$

$n \geq 2$.

$$\text{(I)} \qquad \fint_{\partial D} u \, d\sigma = u(x_0) \qquad \forall u \in H(D) \cap C^1(\overline{D})$$

$$\Updownarrow$$

D = Euclidean ball centered at x_0

- "$\Uparrow$" Gauss
- "$\Downarrow$" Fichera (1985)

- <u>OPEN PROBLEM</u>: Stability of (I)

(II) $$\fint_D u\,dx = \fint_{\partial D} u\,d\sigma \qquad \forall u \in H(D) \cap C^1(\overline{D})$$

$$\Updownarrow$$

D = Euclidean ball <u>centered at a point of D</u>

- "$\Uparrow$" Gauss
- "$\Downarrow$" Fichera (1985)
- <u>OPEN PROBLEM</u>: stability for (II).