Strong Sobolev instability of quasi-periodic solutions of the 2D cubic Schrödinger equation

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joint work with
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The defocusing cubic NLS

- Consider the equation

$$-iu_t + \Delta u = |u|^2 u$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.

- Conserved quantities: the Hamiltonian and the mass

$$E[u] = \int_{\mathbb{T}^2} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 \right) dx$$

$$\mathcal{M}[u] = \left(\int_{\mathbb{T}^2} |u|^2 dx \right)^{\frac{1}{2}}.$$

Two very related problems

- Transfer of energy.
- Lyapunov (in)stability of invariant objects of the dynamical system.

Transfer of energy

- Fourier series of u,

$$u(x,t) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{inx}.$$

- Can we have transfer of energy to high modes as $t \to +\infty$?
- We measure it with the growth of s-Sobolev norms ($s > 1$)

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n \in \mathbb{Z}^2} \langle n \rangle^{2s} |a_n(t)|^2 \right)^{1/2},$$

where $\langle n \rangle = (1 + |n|^2)^{1/2}$.

- Thanks to mass and energy conservation,

$$\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C \|u(0)\|_{H^1(\mathbb{T}^2)} \quad \text{for all} \quad t \geq 0.$$

Growth of Sobolev norms for $-iu_t + \Delta u = |u|^2 u$

- Cubic defocusing NLS on $\mathbb{T}$ (Zakharov-Shabat eq) is integrable: no growth of Sobolev norms.
- Kuksin (1997): growth of Sobolev norms starting from an already large initial data for NLS on $\mathbb{T}^2$.

Theorem (I-team: Colliander, Keel, Staffilani, Takaoka, Tao (2010))

Fix $s > 1$, $K \gg 1$ and $\delta \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0)\|_{H^s} \leq \delta, \qquad \|u(T)\|_{H^s} \geq K.$$

- The result also applies to $s \in (0,1)$ (backward energy cascade)

More results

- M. Guardia and V. Kaloshin: $T \sim e^{\left(\frac{K}{\delta}\right)^{\beta}}$ for some $\beta > 1$.
- M. Guardia, E. H. and M. Procesi: generalization to

$$-iu_t + \Delta u = |u|^{2p} u \qquad \text{with } p \in \mathbb{N}.$$

- Bourgain conjecture: Take $s > 1$. There exists a solution such that

$$\limsup_{t \to +\infty} \|u(t)\|_{H^s} = +\infty.$$

- Z. Hani, B. Pausader, N. Tzvetkov, N. Visciglia: unbounded growth for the cubic NLS with $x \in \mathbb{R} \times \mathbb{T}^2$.
- The conjecture remains open in all compact manifolds.

Dynamical systems point of view

$$-iu_t + \Delta u = |u|^2 u$$

- $u = 0$ is an elliptic critical point: linearly stable in any H^s topology
- It is Lyapunov stable or unstable? It depends on the topology.
- Stability in L^2 and H^1 topology.
- I-team result $\Rightarrow$ Instability in the H^s topology, $s > 0$, $s \neq 1$.

Instability of other invariant objects?

- What about the stability/instability of other invariant objects of the cubic NLS?
- For which time ranges?
- Can we attain Sobolev norm explosion starting arbitrarily close to an invariant object?

Hani approach towards the proof of Bourgain conjecture

- Bourgain conjecture: There exists a solution such that

$$\limsup_{t\to+\infty} \|u(t)\|_{H^s} = +\infty$$

- Long-time strong instability: Let $(X, \|\cdot\|)$ be a Banach space and Φ^t a flow on it. We say Φ^t exhibits long-time strong instability near $u \in X$ if for every $\delta \in (0,1)$ and $K > 1$, there exists $u' \in B_\delta(u)$ such that $\sup_{t>0} \|\Phi^t(u')\| > K$.
- Assume there exists $\emptyset \neq \mathcal{D} \subset \mathcal{F} \subset X$ with
  - $\mathcal{F}$ closed in X
  - $\mathcal{D}$ dense in $\mathcal{F}$.
  - All u in $\mathcal{D}$ have long-time strong instability within $\mathcal{F}$ (i.e. $u' \in \mathcal{F}$)
- Then, there exists $u^* \in \mathcal{F}$ such that $\limsup_{t\to+\infty} \|\Phi^t(u^*)\| = +\infty$.

- I-team result: long-time strong instability of the solution $u \equiv 0$ in H^s for all $s \in (0,1) \cup (1,\infty)$

Stability/Instability of invariant objects: plane waves

- Plane waves (periodic orbits): $u(t,x) = Ae^{i(mx-\omega t)}$ with $\omega = m^2 + A^2$.
- Stability result (Faou, Gauckler and Lubich 2013): Fix $N > 1$. There exists $s_0 > 0$ such that "many" plane waves satisfy the following:

  For any $s \geq s_0$, $\delta \ll 1$ and any initial condition $u_0(x)$ satisfying $\|u_0(x) - Ae^{imx}\|_{H^s} \leq \delta$, its orbit satisfies

$$\|u(t,x) - Ae^{i(mx-\omega t)}\|_{H^s} \lesssim \delta \quad \text{for} \quad t \lesssim \delta^{-N}.$$

- Many: For any m and a full measure set of A

Stability/Instability of invariant objects: plane waves

$$u(t,x) = Ae^{i(mx-\omega t)}, \qquad \omega = m^2 + A^2$$

- Instability result (Hani 2011): Fix $s \in (0,1)$, $K \gg 1$ and $\delta \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0) - Ae^{imx}\|_{H^s} \leq \delta, \qquad \|u(T)\|_{H^s} \geq K.$$

- Remark: Instabilities are only proved for $s \in (0,1)$.

Transfer of energy close to invariant tori

- Goal: Transfer of energy close to invariant (quasiperiodic) tori.
- We look at the "simplest" tori: 1D tori (finite gap solutions).
- 1D Cubic NLS $i\partial_t q = -\partial_{xx} q + |q|^2 q$, $x \in \mathbb{T}$, is integrable.
- It admits global Birkhoff coordinates:

$$\begin{aligned} \Phi : L^2(\mathbb{T}) &\longrightarrow \ \ell^2 \times \ell^2 \\ q \quad &\longmapsto \ (z_m, \bar{z}_m)_{m \in \mathbb{Z}}, \end{aligned}$$

  such that 1D cubic NLS becomes $i\dot{z}_m = \alpha_m(I) z_m$ where $I_m = |z_m|^2$ are the actions.
- All solutions are periodic/quasi-periodic/almost-periodic.

1D invariant tori

- Consider one of the $\mathtt{d}$-dimensional quasiperiodic tori:
  - Excited "modes" $\mathcal{S}_0 = (\mathtt{m}_1, \dots \mathtt{m}_\mathtt{d}) \subset \mathbb{Z}$.
  - Actions vector $I_\mathtt{m} = (I_{\mathtt{m}_1}, \dots, I_{\mathtt{m}_\mathtt{d}}) \in \mathbb{R}_+^\mathtt{d}$.
- Define

$$\mathtt{T}^\mathtt{d} = \left\{ (z_m)_{m \in \mathbb{Z}} : |z_{\mathtt{m}_i}|^2 = I_{\mathtt{m}_i}, i = 1, \dots, \mathtt{d},\ z_m = 0 \text{ if } m \notin \mathcal{S}_0 \right\}.$$

- Lyapunov stable as invariant objects of 1D cubic NLS.
- Consider them as invariant objects for the 2D equation.
- For what time scales is this torus Lyapunov stable/unstable?
- H^s norm explosion from a small neighborhood of it?

Stability of the finite gaps solution

Theorem (A. Maspero – M. Procesi)

Fix $s > 0$. For a generic choice of support sites $\mathcal{S}_0$ there exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a positive measure set $\mathcal{I} \subset (0, \varepsilon)^{\mathtt{d}}$ such that the following holds true for any torus $\mathtt{T}^{\mathtt{d}} = \mathtt{T}^{\mathtt{d}}(\mathcal{S}_0, I_{\mathtt{m}})$ with $I_{\mathtt{m}} \in \mathcal{I}$:

Fix any $\delta \ll 1$. Then, any solution of cubic defocusing NLS $u(t)$ such that

$$\mathrm{dist}_{H^s}\left(u(0), \mathtt{T}^{\mathtt{d}}\right) \le \delta \quad \textit{and} \quad \left||z_m|^2 - I_m^0\right| \le \delta^2 \quad \textit{for all } m \in \mathcal{S}_0$$

satisfies

$$\mathrm{dist}_{H^s}\left(u(t), \mathtt{T}^{\mathtt{d}}\right) \lesssim \delta \qquad \textit{for all } |t| \lesssim \delta^{-2}.$$

Some comments

- ε is small $\Rightarrow$ the tori are small.
- Generic choice of $\mathcal{S}_0$: The modes in $\mathcal{S}_0$ cannot satisfy a finite number of relations (number of relations only depending on $\mathtt{d}$).
- The set of "good" actions $\mathcal{I} \subset (0, \varepsilon)^{\mathtt{d}}$ has positive relative measure (but not asymptotically full).
- For actions in some positive measure set in $(0, \varepsilon)^{\mathtt{d}}$ the tori become hyperbolic and for another positive measure set they are elliptic.

Main result

Theorem (M. Guardia – Z. Hani – E. H. – A. Maspero – M. Procesi)

Fix $s \in (0,1)$. For a generic choice of support sites $\mathcal{S}_0$ there exists ε_0, such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists an asymptotically full measure set $\mathcal{I}' \subset (0,\varepsilon)^{\mathtt{d}}$ such that the following holds true for any torus $\mathtt{T}^{\mathtt{d}} = \mathtt{T}^{\mathtt{d}}(\mathcal{S}_0, I_{\mathtt{m}})$ with $I_{\mathtt{m}} \in \mathcal{I}'$:
For any $\delta \ll 1$ and $K \gg 1$ there exists an orbit of cubic defocusing NLS and a time T such that

- $\mathrm{dist}_{H^s}\left(u(0), \mathtt{T}^{\mathtt{d}}\right) \leq \delta$
- $\|u(T)\|_{H^s} \geq K.$
- $|T| \lesssim e^{\left(\frac{K}{\delta}\right)^{\beta}}$ *for some $\beta > 1$ independent of K, δ.*

A result with $s > 1$

Theorem (M. Guardia – Z. Hani – E. H. – A. Maspero – M. Procesi)

Choose $\mathcal{S}_0$ and $I_{\mathtt{m}}$ as before. Fix $s > 1$. For any $K \gg 1$ there exists an orbit of cubic defocusing NLS and a time T such that

- $\mathrm{dist}_{L^2}\left(u(0), \mathtt{T}^{\mathtt{d}}\right) \leq K^{-\sigma}$
- $\|u(T)\|_{H^s} \geq K\|u(0)\|_{H^s}$.
- $|T| \lesssim e^{K^{\sigma'}}$.

($\sigma, \sigma' > 1$ are independent of K)

Comments

- $s \in (0,1)$ implies that the obtained solution has very small mass but very large energy. (inverse cascade of energy)
- Transversal instability: these tori are stable as solutions of 1D NLS but there are 2D solutions arbitrarily close to the torus which attain large Sobolev norms.
- Same result is valid for NLS on $\mathbb{T}^N$ with $N \geq 2$.
- $\mathcal{I} \subset \mathcal{I}'$: all tori satisfying Maspero-Procesi result (elliptic tori) are unstable after longer time scales.
- $\mathcal{I}'$ has relative measure as close to 1 as desired.

Comments

- We can achieve such growth for both elliptic and partially hyperbolic tori.
- Even if this hyperbolicity is in principle "good" for instabilities, it is not clear how to use it to achieve growth of Sobolev norms (it only involves few Fourier modes).
- To carry on our process, we need to kill the hyperbolic directions.
- We restrict to invariant subspaces (invariant sublattices of $\mathbb{Z}^2$) where the torus is elliptic.

Sketch of the proof: Analysis of the linearization

- Express 1D NLS in Birkhoff coordinates
- Asymptotic estimates for the 1D eigenvalues of the linearization around the torus as $|m| \to \infty$.
- Procesi-Maspero: Reduce the linearization of 2D NLS around T^{d} to constant coefficients (KAM scheme).
- Asymptotic estimates for the 2D eigenvalues of the linearization around the torus: For $n = (n_1, n_2) \in \mathbb{Z}^2$,

$$\Omega_n = \left(n_1^2 + n_2^2 + \mathcal{M}^2 + \frac{\mathcal{O}(\varepsilon^2)}{\langle n_1 \rangle^2} \right)$$

  for $n_2 \neq 0$

- Eigenvalues are very close to resonance for $|n_1| \gg 1$.

Sketch of the proof: Birkhoff normal form

- Procesi-Maspero: Birkhoff normal form to eliminate completely the quadratic terms in the equation.
- Birkhoff normal form to eliminate cubic non-resonant terms.
- I-team approach: Use cubic resonant terms to achieve instabilities.

The I-team approach for solutions close to $u = 0$

$$-iu_t + \Delta u = |u|^2 u$$

- Cubic NLS as an ODE (of infinite dimension) for the Fourier coefficients of u:

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a_{n_2}} a_{n_3}, \qquad n \in \mathbb{Z}^2.$$

- Drift through resonances.
- Resonant monomial

$$n_1 - n_2 + n_3 - n = 0 \quad \text{and} \quad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0$$

- Non-degenerate resonances form a rectangle in $\mathbb{Z}^2$.

The I-team approach for solutions close to $u = 0$

- Select (cleverly) a finite set of modes $\Lambda \subset \mathbb{Z}^2$ such that

$$U_\Lambda = \{a_n = 0 : n \notin \Lambda\}$$

is invariant under the (degree 4) Birkhoff normal form of NLS.

$$-i\dot{a}_n = |n|^2 a_n + \sum_{n_1, n_2, n_3, n \text{ form a rectangle}} a_{n_1} \overline{a_{n_2}} a_{n_3}, \qquad n \in \mathbb{Z}^2.$$

- Size of Λ depends on the desired growth of Sobolev norms.
- Use combinatorics to choose the modes in Λ such that the dynamics on U_Λ is "easy" to analyze.

The I-team approach for the cubic case

- Choosing well Λ and after several reductions (taking advantage strongly of the particular form of NLS):

  Finite dimensional (toy) model

  $$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 1, \ldots N.$$

  gives the dynamics on Λ.

- Each b_j represents "many" modes in Λ.
- Hamiltonian system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.
- Growth of Sobolev norms $\equiv$ Pushing energy from b_1 to b_N.

The toy model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0, \dots N,$$

- It possesses a sequence of periodic orbits connected through non-transversal heteroclinic orbits.
- First periodic orbit $b_1 \neq 0$, $b_j = 0$ for $j \neq 1$.
- Last periodic orbit $b_N \neq 0$, $b_j = 0$ for $j \neq N$.
- I-team: Shadowing these objects gives the desired orbits.

Adapting the I-team approach for solutions close a 1D torus

- After normal form we have an eq. "similar" to the Birkhoff normal form of the original NLS around $u = 0$.
- I-team model on Λ:

$$-i\dot{a}_n = |n|^2 a_n + \sum_{n_1,n_2,n_3 \in \Lambda \text{ form a rectangle with } n} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n \in \Lambda.$$

- Model on a (well chosen) Λ in our setting:

$$-i\dot{a}_n = \Omega_n a_n + \sum_{n_1,n_2,n_3 \in \Lambda \text{ form a rectangle with } n} C_{n_1,n_2,n_3,n} a_{n_1}\overline{a_{n_2}}a_{n_3}, \quad n \in \Lambda.$$

Adapting the I-team approach for solutions close to a 1D torus

$$-i\dot{a}_n = |n|^2 a_n + \sum_{n_1,n_2,n_3 \in \Lambda \text{ form a rectangle with } n} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n \in \Lambda.$$

$$-i\dot{a}_n = \Omega_n a_n + \sum_{n_1,n_2,n_3 \in \Lambda \text{ form a rectangle with } n} C_{n_1,n_2,n_3,n} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n \in \Lambda.$$

Differences:

- Eigenvalues Ω_n are not $|n|^2$, but close to it for modes in Λ.
- Coefficients $C_{n_1,n_2,n_3,n}$ are not $\equiv 1$.
- We drift along almost-resonances.
- This creates extra error terms that blow up for large times.

Adapting the I-team approach

$$-i\dot{a}_n = |n|^2 a_n + \sum_{n_1,n_2,n_3\in\Lambda \text{ form a rectangle with } n} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\Lambda.$$

$$-i\dot{a}_n = \Omega_n a_n + \sum_{n_1,n_2,n_3\in\Lambda \text{ form a rectangle with } n} C_{n_1,n_2,n_3,n}a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\Lambda.$$

- Crucial point for the reduction to the toy model in I-team result: Coefficient of all monomials $= 1$.
- Now: coefficients $\neq 1$ (they depend on the modes).
- This comes from interaction between torus modes/modes in Λ.
- Goal: Choose $\Lambda \subset \mathbb{Z}^2$ to "minimize" this interaction.
- For a well chosen $\Lambda \subset N\mathbb{Z}\times N\mathbb{Z}$, the coefficients of the monomials are $C_{n_1,n_2,n_3,n} = 1 + \mathcal{O}(N^{-1})$ thanks to:
  - Smoothing properties of the reducibility transf. (on $N\mathbb{Z}\times N\mathbb{Z}$).
  - Combinatorial analysis of the Birkhoff NF changes of coord.

Adapting the I-team approach for solutions close a 1D torus

- Treating the extra terms as errors:
  - One can carry on the I-team approach.
  - Estimate the time of diffusion by applying toy model results by M. Guardia – V. Kaloshin.
- This leads to growth of Sobolev norms after incorporating the errors and undoing all the changes of coordinates.

Why $s \in (0,1)$?

- The resonances corresponding to rectangles $\subset \mathbb{Z}^2$ are only approximate, not exact
- $\Omega_n = |n|^2 + \mathcal{M}^2 + \dfrac{\mathcal{O}(\varepsilon^2)}{\langle n_1 \rangle^2}$
- Scaling the set Λ by a factor $J \gg 1$ one has on rectangles

$$\Gamma := \Omega_{n_1} - \Omega_{n_2} + \Omega_{n_3} - \Omega_{n_4} = \mathcal{O}(J^{-2})$$

- Essentially, we approximate $e^{it\Gamma} \sim 1$, which requires $J^{-2} T \ll 1$
- Toy model scaling $b(t) \rightsquigarrow \nu b(\nu^2 t) \Rightarrow T = \mathcal{O}(\nu^{-2}) \implies J\nu \gg 1$
- Small initial Sobolev norm $J^s \nu \ll 1$
- This is possible ONLY IF $s < 1$

Attempt of generalization to 2D tori (work in progress with A. Maspero and M. Procesi)

- 1D finite-gap tori are very special
- What about instability of more general 2D tori?
- Main difference: recall the asymptotic expansion for the finite-gap eigenvalues

$$\Omega_n = \left(n_1^2 + n_2^2 + \mathcal{M}^2 + \frac{\mathcal{O}(\varepsilon^2)}{\langle n_1 \rangle^2} \right)$$

- For more general tori, the decay in the constant direction n_1 has to be replaced by decay in a direction that varies with n
- Related to quasi-Töplitz matrices and to Bourgain's cluster decomposition of the spectrum of $-\Delta$ on the torus (recently, Bambusi-Langella-Montalto: spectrum of $(-\Delta)^\alpha + V$ on the torus)

Thank you!

Thank you for your attention!