On spectral stability of

Aharonov-Bohm operators with moving poles

Joint works with L. Abatangelo, L. Hillairet, C. Léna, B. Noris, M. Nys

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Aharonov–Bohm potential

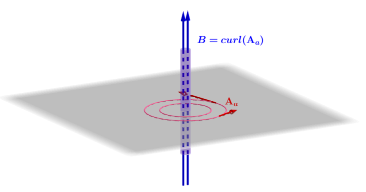
For $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^2$, the Aharonov-Bohm magnetic potential with pole $\boldsymbol{a}$ and circulation $\gamma \in \mathbb{R} \setminus \mathbb{Z}$ is

$$\mathbf{A}_{\boldsymbol{a}}(x_1, x_2) = \gamma \left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right).$$

$\mathbf{A}_{\boldsymbol{a}}$ generates the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ with pole $\boldsymbol{a}$ and circulation γ.

Aharonov–Bohm potential

The AB magnetic field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1, x_2) at the point $\boldsymbol{a}$, as the radius of the solenoid goes to zero and the magnetic flux remains constantly equal to γ.



Negletting the irrelevant coordinate along the solenoid, the problem becomes 2-dimensional.

Aharonov-Bohm effect [Aharonov-Bohm, Phys. Rev. (1959)]

The AB magnetic field is a δ-like magnetic field: a quantum particle moving in $\mathbb{R}^2 \setminus \{\boldsymbol{a}\}$ is affected by the magnetic potential, despite being confined to a region in which the magnetic field is zero.

Aharonov–Bohm potential

The Schrödinger operators with AB vector potential:

$$(i\nabla + \mathbf{A}_{\boldsymbol{a}})^2 u = -\Delta u + 2i\mathbf{A}_{\boldsymbol{a}} \cdot \nabla u + |\mathbf{A}_{\boldsymbol{a}}|^2 u.$$

In $\Omega \subset \mathbb{R}^2$ bounded, open and simply connected, $\forall\, \boldsymbol{a} \in \overline{\Omega}$ the eigenvalue problem

$$\begin{cases} (i\nabla + A_{\boldsymbol{a}})^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a sequence of real diverging eigenvalues $\{\lambda_k(\boldsymbol{a})\}_{k\geq 1}$

$$\lambda_1(\boldsymbol{a}) \leq \lambda_2(\boldsymbol{a}) \leq \cdots \leq \lambda_k(\boldsymbol{a}) \leq \ldots$$

If $\boldsymbol{a} \in \partial\Omega$, then

$$\lambda_k(\boldsymbol{a}) = \lambda_k = \text{eigenvalue of the standard Dirichlet Laplacian.}$$

The case $\gamma = \frac{1}{2}$, half-integer circulation

$$\mathbf{A}_{\boldsymbol{a}}(x) = \mathbf{A_0}(x - \boldsymbol{a}), \quad \text{where} \quad \mathbf{A_0}(x_1, x_2) = \frac{1}{2}\left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$

- Bonnaillie-Noël, Helffer, Hoffmann-Ostenhof [J. Phys. A (2009)] Noris, Terracini [Indiana Univ. Math. J. (2010)]:
  nodal domains of eigenfunctions are related to spectral minimal partitions of the Dirichlet Laplacian with points of odd multiplicity

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- Bonnaillie-Noël, Noris, Nys, Terracini [Analysis and PDE (2014)] Léna [J. Math. Phys. (2015)], Noris, Nys, Terracini [CMP (2015)] Noris, Terracini [Indiana Univ. Math. J. (2010)]: *a strong connection between nodal properties of eigenfunctions and the critical points of the map* $\boldsymbol{a} \mapsto \lambda_j(\boldsymbol{a})$.

Properties of the map $a \mapsto \lambda_j(a)$

Bonnaillie-Noël, Noris, Nys, Terracini (2014), Léna (2015):

- $\forall\, j \geq 1$, the map $\boldsymbol{a} \mapsto \lambda_j(\boldsymbol{a})$ is continuous in Ω and has a continuous extension on $\overline{\Omega}$ (as $\boldsymbol{a} \to \partial\Omega$, $\lambda_j(\boldsymbol{a}) \to j$-th eigenvalue of $-\Delta$ in Ω)

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- **Improved regularity for simple eigenvalues:**
  if $\boldsymbol{b} \in \Omega$ and $\lambda_N(\boldsymbol{b})$ is simple, then the function $\boldsymbol{a} \mapsto \lambda_N(\boldsymbol{a})$ is analytic in a neighborhood of $\boldsymbol{b}$.

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Bonnaillie-Noël, Noris, Nys, Terracini (2014): the behavior of the eigenvalue $\lambda_N(\boldsymbol{a})$ is strongly related to the structure of the nodal lines of the associated eigenfunction:

$$|\lambda_N(\boldsymbol{a}) - \lambda_N(\mathbf{0})| \leq C|\boldsymbol{a}|^{\frac{k+1}{2}} \quad \text{as } \boldsymbol{a} \to \mathbf{0} \in \Omega,$$

where $k \geq 3$ is the number of nodal lines of the eigenfunction associated to the **simple** eigenvalue $\lambda_N(\mathbf{0})$.

Rate of convergence - Interior point $b = \mathbf{0} \in \Omega$

Let $N \geq 1$ be such that $\lambda_N(\mathbf{0})$ is simple. Thus $\lambda_N(\boldsymbol{a}) \underset{\boldsymbol{a}\to\mathbf{0}}{\to} \lambda_N(\mathbf{0})$.

Let $\varphi_N^{\mathbf{0}}$ be an eigenfunction of $(i\nabla + \mathbf{A_0})^2$ associated to $\lambda_N(\mathbf{0})$

$$\begin{cases} (i\nabla + \mathbf{A_0})^2 \varphi_N^{\mathbf{0}} = \lambda_N^{\mathbf{0}} \varphi_N^{\mathbf{0}}, & \text{in } \Omega, \\ \varphi_N^{\mathbf{0}} = 0, & \text{on } \partial\Omega, \end{cases} \quad \text{such that} \quad \int_\Omega |\varphi_N^{\mathbf{0}}(x)|^2 \, dx = 1.$$

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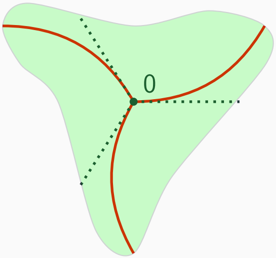
- $\varphi_N^{\mathbf{0}}$ has at 0 a zero or order $\frac{k}{2}$ for some odd $k \in \mathbb{N}$
- $\varphi_N^{\mathbf{0}}$ has got exactly k nodal lines meeting at 0 and dividing the whole angle into k equal parts.

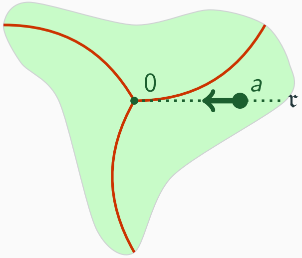


Sharp asymptotics: moving along directions of nodal lines

Theorem [Abatangelo-F., Calc. Var. PDEs 2015]

Let $\mathfrak{r}$ be the half-line tangent to a nodal line of eigenfunction $\varphi_N^{\mathbf{0}}$ associated to $\lambda_N^{\mathbf{0}}$ ending at $\mathbf{0}$. Then, as $\boldsymbol{a} \to \mathbf{0}$ with $\boldsymbol{a} \in \mathfrak{r}$,

$$\frac{\lambda_N(\mathbf{0}) - \lambda_N(\boldsymbol{a})}{|\boldsymbol{a}|^k} \to -4\,\frac{|\beta_1|^2 + |\beta_2|^2}{\pi}\,\mathfrak{m}.$$



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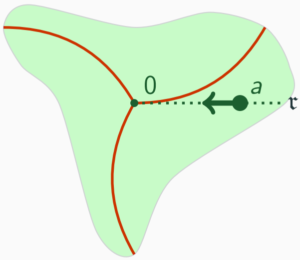


Here

- $(\beta_1, \beta_2) \neq (0,0)$ is s.t.
  $r^{-\frac{k}{2}}\varphi_N^{\mathbf{0}}(r(\cos t, \sin t)) \to \beta_1 \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}} \cos\left(\frac{k}{2}t\right) + \beta_2 \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}} \sin\left(\frac{k}{2}t\right)$ as $r \to 0^+$
- $\mathfrak{m} < 0$ is a negative constant depending only on k.

The constant $\mathfrak{m}$

- **Abatangelo-F., Calc. Var. PDEs 2015**: variational characterization

$$\mathfrak{m} = -\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla w(x)|^2\,dx < 0$$

where w is the unique finite energy weak solution to

$$\begin{cases} -\Delta w = 0, & \text{in } \mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0)\}, \\ w = 0, & \text{on } s := [1,+\infty)\times\{0\}, \\ \frac{\partial w}{\partial \nu}(x_1,0) = \frac{k}{2}x_1^{\frac{k}{2}-1}, & \text{on } \partial\mathbb{R}^2_+ \setminus s, \end{cases}$$

- **Abatangelo-F.-Léna 2018**: $\mathfrak{m} = -\frac{k\pi}{2^{2k+1}}\binom{k-1}{\lfloor\frac{k-1}{2}\rfloor}^2$

Blow-up

It is not restrictive to assume that $\beta_1 = 0$ *(rotate the axes so that the positive* x_1*-axis is tangent to one of the* k *nodal lines of* $\varphi_N^{\mathbf{0}}$ *ending at* $\mathbf{0}$*).*

Theorem

$$\frac{\varphi_N^{\boldsymbol{a}}(|\boldsymbol{a}|x)}{|\boldsymbol{a}|^{k/2}} \to \frac{\beta_2}{\sqrt{\pi}}\Psi \quad \text{as } \boldsymbol{a} = (|\boldsymbol{a}|, 0) \to 0,$$

in some suitable Sobolev norm, a.e. and in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{\mathbf{e}\}, \mathbb{C})$, where $\mathbf{e} = (1, 0)$.

Ψ is the unique function (with local finite "energy") satisfying

$$\begin{cases} (i\nabla + A_{\mathbf{e}})^2\Psi = 0, & \text{in } \mathbb{R}^2, \\ \Psi \sim \psi_k \text{ (up to suitable phases) near } \infty, \end{cases}$$

where $\psi_k(r\cos t, r\sin t) = r^{k/2}\sin\left(\frac{k}{2}\, t\right)$.

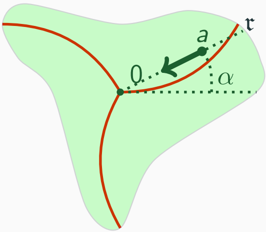
Moving along any direction

Theorem [Abatangelo-F. SIAM J. Math. Anal. (2016)]

If $\alpha \in [0, 2\pi)$, then, as $\boldsymbol{a} = |\boldsymbol{a}|(\cos\alpha, \sin\alpha) \to \mathbf{0}$,

$$\frac{\lambda_N(\mathbf{0}) - \lambda_N(\boldsymbol{a})}{|\boldsymbol{a}|^k} \to C_0 \cos(k(\alpha - \alpha_0))$$

with $C_0 = -4\frac{|\beta_1|^2+|\beta_2|^2}{\pi}\mathfrak{m}$.



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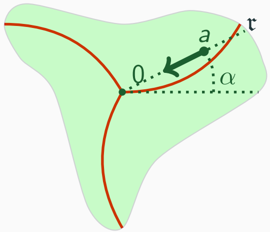


The leading term in the Taylor expansion

$$\lambda_N(\mathbf{0}) - \lambda_N(\boldsymbol{a}) = P(\boldsymbol{a}) + o(|\boldsymbol{a}|^k), \quad \text{as } |\boldsymbol{a}| \to 0^+,$$

is then given by $P(|\boldsymbol{a}|(\cos\alpha, \sin\alpha)) = C_0|\boldsymbol{a}|^k \cos(k(\alpha - \alpha_0))$.

Hence $P(a_1, a_2) = C_0\Re\big(e^{-ik\alpha_0}(a_1 + i\, a_2)^k\big)$ and the polynomial P is *harmonic*.

The importance of being $1/2$

The special features of A.-B. operators with circulation $\frac{1}{2}$ (or $\gamma \in \frac{\mathbb{Z}}{2}$) played a crucial role in the previous results.

- Local energy estimates for eigenfunctions near the limit pole are performed by studying an Almgren type quotient; for $\gamma = \frac{1}{2}$ this can be estimated using a representation formula by Green's functions for solutions to the corresponding Laplace problem on the twofold covering.
- The limit profile is constructed using the fact that it vanishes on the special directions determined by the nodal lines of limit eigenfunctions $\rightsquigarrow$ sharp relation between the asymptotics of the eigenvalue function and the number of nodal lines (which is strongly related to the order of vanishing of the limit eigenfunction).

The case $\gamma \in \mathbb{R} \setminus \frac{\mathbb{Z}}{2}$

- A reduction to the Laplacian on the twofold covering manifold is no more available.
- Magnetic eigenfunctions vanish at the pole a but they do not have nodal lines ending at a.

To derive sharp energy estimates (through Almgren frequency function) we need to give a detailed description of the behaviour of eigenfunctions at the pole and study the dependence of the coefficients of their asymptotic expansion with respect to the moving pole a.

Non-half-integer circulation

Let us consider the AB vector potential with circulation $\gamma \notin \frac{\mathbb{Z}}{2}$

$$\mathbf{A}_{\boldsymbol{a}}(x_1, x_2) = \gamma\left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right).$$

Assume that $\exists\, N \geq 1$ such that $\lambda_N(\mathbf{0})$ is simple. Thus $\lambda_N(\boldsymbol{a}) \overset{\boldsymbol{a}\to\mathbf{0}}{\to} \lambda_N(\mathbf{0})$.

If $\varphi_N^{\mathbf{0}}$ is an eigenfunction of $(i\nabla + \mathbf{A}_{\mathbf{0}})^2$ associated to $\lambda_N^{\mathbf{0}}$, then by [F., Ferrero, Terracini (2011)] we know that

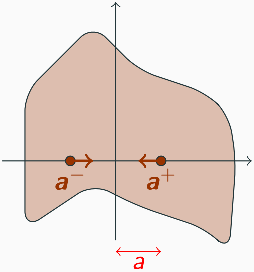
$\varphi_{\mathbf{0}}^N$ vanishes at $\mathbf{0}$ with a vanishing order equal to $|\gamma - k|$ for some $k \in \mathbb{Z}$

Theorem [Abatangelo-F.-Noris-Nys, Analysis PDEs (2018)]

$$|\lambda_N(\boldsymbol{a}) - \lambda_N(\mathbf{0})| = O\left(|\boldsymbol{a}|^{1+\lfloor 2|\gamma-k|\rfloor}\right) \quad \text{as } |\boldsymbol{a}| \to 0.$$

Aharonov-Bohm operators with two colliding poles

For $a > 0$, let
$\boldsymbol{a}^- = (-a, 0)$ and $\boldsymbol{a}^+ = (a, 0)$
be the poles of the AB potential



$$\mathbf{A}_{\boldsymbol{a}^-,\boldsymbol{a}^+}(x) := -\mathbf{A}_{\boldsymbol{a}^-}(x) + \mathbf{A}_{\boldsymbol{a}^+}(x)$$
$$= -\frac{1}{2}\frac{(-x_2, x_1 + a)}{(x_1 + a)^2 + x_2^2} + \frac{1}{2}\frac{(-x_2, x_1 - a)}{(x_1 - a)^2 + x_2^2}.$$

Aharonov-Bohm operators with two colliding poles

$$\mathbf{A}_{a^-,a^+}(x) := -\mathbf{A}_{a^-}(x) + \mathbf{A}_{a^+}(x)$$

Let $\Omega \subseteq \mathbb{R}^2$ be open, bounded and connected with $0 \in \Omega$.

Let $\{\lambda_k^a\}_{k\geq 1}$ be the eigenvalues of $(i\nabla + \mathbf{A}_{a^-,a^+})^2$ in Ω with homogenous Dirichlet boundary conditions.

Let $\{\lambda_k\}_{k\geq 1}$ be the eigenvalues of the Dirichlet Laplacian $-\Delta$ in Ω.

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Theorem [Léna, J. Math. Physics (2015)]

For every $k \geq 1$,

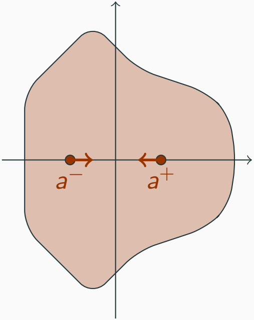
$$\lim_{a\to 0} \lambda_k^a = \lambda_k.$$

Problem: sharp asymptotics for the eigenvalue variation $\lambda_k^a - \lambda_k$ as the two poles $\boldsymbol{a}^-, \boldsymbol{a}^+$ coalesce towards a point?

Symmetric case

Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma(x_1, x_2) = (x_1, -x_2)$. Let Ω be such that

$$\sigma(\Omega) = \Omega \quad \text{and} \quad 0 \in \Omega.$$



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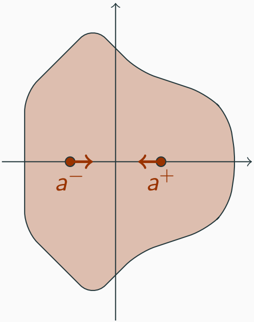


Let λ_N be a simple eigenvalue of the Dirichlet Laplacian on Ω and φ_N be an associated $L^2(\Omega)$-normalized eigenfunction

$\rightsquigarrow \exists\, k \in \mathbb{N}$, $\beta \neq 0$, $\alpha \in [0, \pi)$ s.t.

$$\varphi_N(r(\cos t, \sin t)) \underset{r \to 0^+}{\sim} \beta\, r^k \sin(\alpha - k t)$$

- If $k = 0$, φ_N does not vanish near 0 and $\beta \sin \alpha = \varphi_N(0)$.
- $k = 1$: 0 is a regular point in the nodal set of φ_N and $\beta^2 = |\nabla \varphi_N(0)|^2$.
- If $k \geq 1$, the nodal set of φ_N near 0 consists of $2k$ regular half-curves meeting at 0 with equal angles; the minimal slope of half-curves is $\frac{\alpha}{k}$.

Symmetric case

Symmetry (and simplicity of λ_N) $\rightsquigarrow \varphi_N$ is either $\underset{\alpha=\frac{\pi}{2}}{\underset{\uparrow}{\text{even}}}$ or $\underset{\alpha=0}{\underset{\uparrow}{\text{odd}}}$ in x_2

Theorem [Abatangelo-F.-Hillairet-Léna, J. Spectr. T. (2019)]

Let φ_N be even in x_2. Then

$$\text{if } k=0, \quad \lambda_N^a = \lambda_N + \frac{2\pi|\varphi_N(0)|^2}{|\log a|} + o\left(\frac{1}{|\log a|}\right), \qquad \text{as } a \to 0^+,$$

$$\text{if } k\geq 1, \quad \lambda_N^a = \lambda_N + \frac{k\pi\beta^2}{4^{k-1}}\binom{k-1}{\lfloor\frac{k-1}{2}\rfloor}^2 a^{2k} + o\left(a^{2k}\right), \quad \text{as } a \to 0^+.$$

Theorem [Abatangelo-F.-Léna, ESAIM COCV, to appear]

Let φ_N be odd in x_2. Then

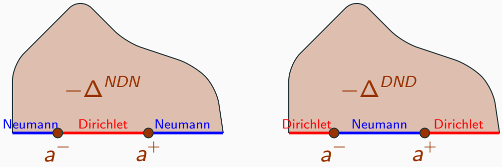
$$\lambda_N^a = \lambda_N - \frac{k\pi\beta^2}{4^{k-1}}\binom{k-1}{\lfloor\frac{k-1}{2}\rfloor}^2 a^{2k} + o\left(a^{2k}\right), \quad \text{as } a \to 0^+.$$

Idea of the proof.

1. Isospectrality. The sequence $\{\lambda_k^{a}\}_{k\geq 1}$ is the union, counted with multiplicities, of sequences $\{\lambda_k^{NDN}(\boldsymbol{a}^+, \boldsymbol{a}^-)\}_{k\geq 1}$, $\{\lambda_k^{DND}(\boldsymbol{a}^+, \boldsymbol{a}^-)\}_{k\geq 1}$.

$$\{\lambda_k^{NDN}(\boldsymbol{a}^+, \boldsymbol{a}^-)\}_{k\geq 1} = \left\{ \begin{array}{l} \text{eigenvalues of Neumann-Dirichlet-Neumann} \\ \text{Laplacian } -\Delta^{NDN} \text{ on the half domain} \end{array} \right\}$$

$$\{\lambda_k^{DND}(\boldsymbol{a}^+, \boldsymbol{a}^-)\}_{k\geq 1} = \left\{ \begin{array}{l} \text{eigenvalues of Dirichlet-Neumann-Dirichlet} \\ \text{Laplacian } -\Delta^{DND} \text{ on the half domain} \end{array} \right\}$$



See **[Bonnaillie-Noël-Helffer-Hoffmann-Ostenhof, J. Phys. A (2009)]** for isospectrality results for a single pole.

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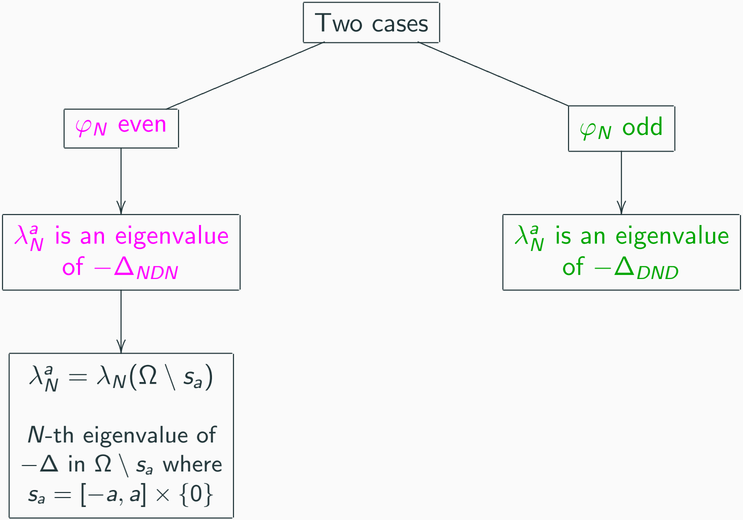
2.



Even case

In the even case, the problem reduces to the study of the asymptotics of

$$\lambda_N(\Omega \setminus s_a) \quad \text{as } a \to 0^+.$$

- **Courtois [J. Funct. Anal., 1995]**:

$$\lambda_N(\Omega \setminus s_a) = \lambda_N + \mathrm{Cap}_\Omega(s_a, \varphi_N) + o\left(\mathrm{Cap}_\Omega(s_a, \varphi_N)\right),$$

as $a \to 0^+$, where $\mathrm{Cap}_\Omega(s_a, \varphi_N) = \inf\limits_{\substack{f \in H_0^1(\Omega) \\ f = \varphi_N \text{ on } s_a}} \int_\Omega |\nabla f|^2$.

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as $a \to 0^+$, where $\mathrm{Cap}_\Omega(s_a, \varphi_N) = \inf_{\substack{f \in H^1_0(\Omega) \\ f = \varphi_N \text{ on } s_a}} \int_\Omega |\nabla f|^2$.

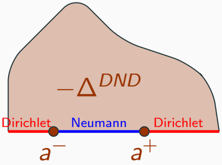
- Sharp estimates of $\mathrm{Cap}_\Omega(s_a, \varphi_N)$ passing to elliptic coordinates

$$\begin{cases} x_1 = a \cosh(\xi) \cos(\eta), \\ x_2 = a \sinh(\xi) \sin(\eta). \end{cases}$$

Odd case

In the odd case, the problem reduces to the study of the asymptotics of

$$\{\lambda_k^{DND}(\boldsymbol{a}^+, \boldsymbol{a}^-)\}_{k\geq 1} = \left\{ \begin{array}{l} \text{eigenvalues of Dirichlet-Neumann-Dirichlet} \\ \text{Laplacian } -\Delta^{DND} \text{ on the half domain} \end{array} \right\}$$



DND problem

Let λ_N be simple and let φ_N be an associate normalized eigenfunction, i.e.

$$\begin{cases} -\Delta\varphi_N = \lambda_N\varphi_N, & \text{in } \Omega, \\ \varphi_N = 0, & \text{on } \partial\Omega, \\ \int_\Omega \varphi_N^2(x)\,dx = 1. \end{cases}$$

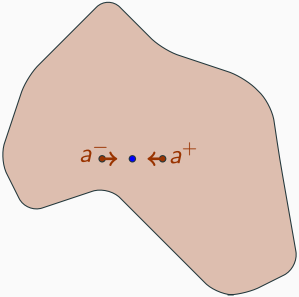
Then there exist $k \in \mathbb{N} \setminus \{0\}$ and $\beta \in \mathbb{R} \setminus \{0\}$ such that

$$\varphi_N(r\cos t, r\sin t) \underset{r\to 0}{\sim} \beta\, r^k \sin(kt).$$

Gadyl'shin (1992), Abatangelo-F.-Léna (2018):

$$\lim_{a\to 0^+} \frac{\lambda_N - \lambda_N(a)}{a^{2k}} = \beta^2\,\frac{k\pi}{2^{2k-1}}\binom{k-1}{\left\lfloor \frac{k-1}{2}\right\rfloor}^2$$

Non symmetric case



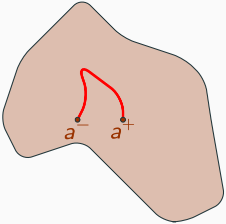
No symmetry assumption on Ω
Let us assume the N-th eigenvalue λ_N of $-\Delta$ in Ω is simple. Let φ_N be a $L^2(\Omega)$-normalized eigenfunction associated to λ_N.

Theorem [Abatangelo-F.-Léna, Advanced Nonlin. Studies (2017)]

If $\varphi_N(0) \neq 0$ (i.e. $k = 0$) then

$$\lambda_N^a - \lambda_N = \frac{2\pi\, \varphi_N^2(0)}{|\log a|}(1 + o(1)) \quad \text{as } a \to 0^+.$$

Idea of the proof.



1. If a is small and φ_N^a is an eigenfunction associated with λ_N^a, then, in a neighborhood of 0, the nodal set of φ_N^a consists in a single regular curve K_a connecting $\boldsymbol{a}^-$ and $\boldsymbol{a}^+$ and concentrating around 0. **[Noris-Terracini (2010), Helffer-Hoffmann-Ostenhof-Hoffmann-Ostenhof-Owen (1999), Alziary-Fleckinger-Pellé-Takáč (2003)]**

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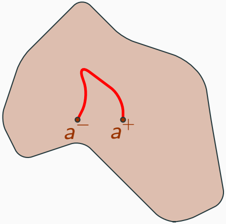

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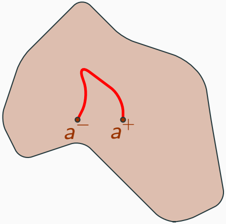

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2. For all $a > 0$ sufficiently small, $\lambda_N^a = \lambda_N(\Omega \setminus K_a)$ (Gauge invariance).

3. We denote as $d_a := \operatorname{diam} K_a$ the diameter of K_a. We already know that

$$\lambda_N(\Omega \setminus K_a) - \lambda_N = \varphi_N^2(0)\frac{2\pi}{|\log d_a|} + o\left(\frac{1}{|\log d_a|}\right), \quad \text{as } a \to 0^+.$$

It remains to estimate d_a, i.e. the diameter of nodal lines of magnetic eigenfunctions near the collision point:

$$\lim_{a \to 0^+} \frac{|\log a|}{|\log d_a|} = 1.$$

Reaching a point on the boundary: $a \to b \in \partial\Omega$

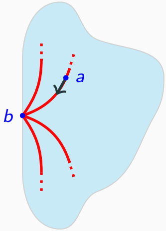
In this case the limit operator is no more singular and the magnetic eigenvalues converge to those of the standard Laplacian: $\lambda_k(\boldsymbol{b}) = \lambda_k$.

Noris, Nys, Terracini (2015):

- if λ_N is simple and its eigenfunction φ_N has at $\boldsymbol{b} \in \partial\Omega$ a zero of order $j \geq 2$ ($j-1$ nodal lines end at $\boldsymbol{b}$) then $\exists C > 0$ s.t.

$$\lambda_N(\boldsymbol{a}) - \lambda_N \leq -C|\boldsymbol{a} - \boldsymbol{b}|^{2j}$$

for $\boldsymbol{a} \to \boldsymbol{b}$ along a nodal line.



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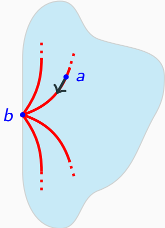
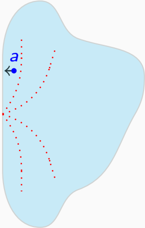


- if λ_N is simple and $\boldsymbol{a}$ approaches a boundary point where no nodal lines of φ_N end, then $\exists C > 0$ s.t.

$$\lambda_N(\boldsymbol{a}) - \lambda_N \geq C(\mathrm{dist}(\boldsymbol{a}, \partial\Omega))^2.$$



Sharp asymptotics at the boundary

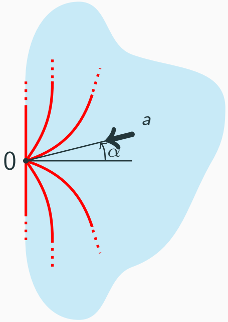
Let $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and
$p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1,\ x_1 > 0\}$.

Theorem [Abatangelo-F-Noris-Nys, JFA (2017)]

There exists $\mathfrak{c}_p \in \mathbb{R}$ such that

$$\frac{\lambda_N - \lambda_N(\boldsymbol{a})}{|\boldsymbol{a}|^{2j}} \to |\beta|^2\, \mathfrak{c}_p, \quad \text{as } \boldsymbol{a} = |\boldsymbol{a}|p \to 0.$$

- the function $p \mapsto \mathfrak{c}_p$ is continuous on $\mathbb{S}^1_+$ and tends to 0 as $p \to (0, \pm 1)$;



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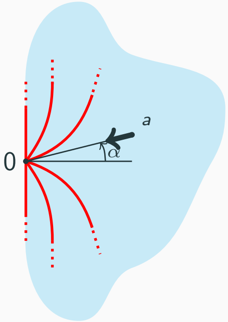

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- $\mathfrak{c}_p < 0$ if the half-line $\{tp : t \geq 0\}$ is the bisector of two nodal lines of φ_N or of one nodal line and the boundary.