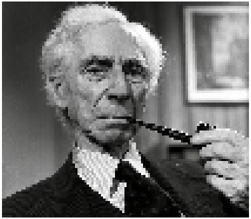
The idea of *infinity* in Mathematics between Science and Philosophy

Variational methods, with applications to problems in mathematical physics and geometry.
Dedicated to Antonio for its 75th birthday.

Vieri Benci
Università di Pisa

November 26, 2019



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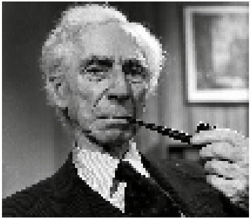

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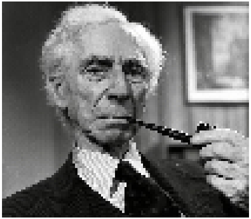
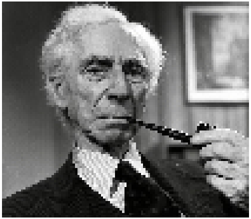


For Bertrand Russell (B. Russell, Mysticism and Logic, 1901) the main problems of the Philosophy of Mathematics are three and all of them are related to the idea of infinity:

1. The problem of infinite numbers;
2. the problem of the continuum;



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2. the problem of the continuum;
3. the problem of the infinitesimal quantities.

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2. the second has been solved by Dedekind by identifying the geometric continuum with the real line (with the Dedekind axiom);
3. the third has been solved by Weierstrass expelling the infinitesimals from the Kingdom of Mathematics.

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- - simplifies computations;
- - allows to build richer models of reality;
- - gives a deeper understanding of the cardinal and ordinal numbers;
- - expands the epistemological horizon of the foundations of Mathematics.

The first philosophical problem: how to count infinite sets.

Let start our discussion with the first problem namely the possibility of "**counting**" the elements of infinite sets.

Hume

Euclides

Let us recall the two fundamental principles which rule the operation of "**counting**".

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- **The Hume principle** - *Two sets have the same number of elements if and only if there exists a biunique correspondence between them.*

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- **The Hume principle** - *Two sets have the same number of elements if and only if there exists a biunique correspondence between them.*
- **The Euclides principle** (5° common notion) - *The whole is greater than the part.*

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Galileo is one among the *natural philosophers* who emphasized this point.

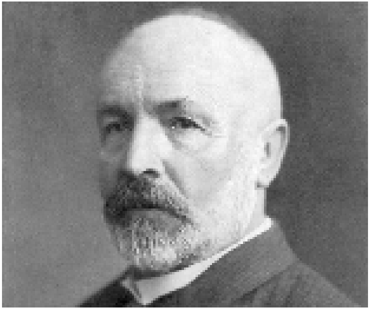
The square numbers are a *part* of all the numbers, but there is a *biunique correspondence* with all the numbers.

Galileo's law

1	$\longleftrightarrow$	1
2	$\longleftrightarrow$	4
3	$\longleftrightarrow$	9
4	$\longleftrightarrow$	16
5	$\longleftrightarrow$	25
6	$\longleftrightarrow$	36
7	$\longleftrightarrow$	49
8	$\longleftrightarrow$	64
9	$\longleftrightarrow$	81
10	$\longleftrightarrow$	100
...	$\longleftrightarrow$	...

$$s = \frac{1}{2} g t^2$$

Cantor



Cantor has been the first to understand that eliminating one of the two principle (namely the Euclides Principle) it is possible to get a strange but consistent theory

Infinite cardinals

Figure: Infinite cardinals

The "quirk" of cardinal numbers is their arithmetic: if $\mathfrak{a}$ and $\mathfrak{b}$ are infinite cardinal numbers, then

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{a} \times \mathfrak{b} = \max(\mathfrak{a}, \mathfrak{b})$$

The ordinal numbers

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Cantor also understood that it is possible to use a different strategy to count sets. And changing strategy with infinite sets, not only you get different results, but also different numbers.
In any case, the ordinal numbers are as weird as the cardinal numbers:

$$\omega + 1 > 1 + \omega = \omega$$

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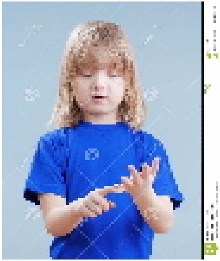
The answer is "*yes*".

Three ways to count



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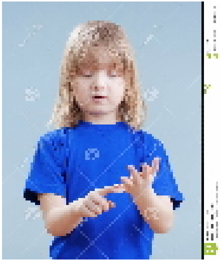

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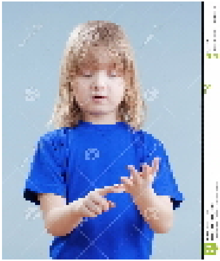

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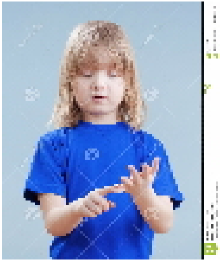

There exist (at least) three ways to count the elements of a set:

1. Child three years old: biunique correspondence.
2. Child five years old: put the items to be counted in a row and then "one, two, three,..."
3. Child ten years old: organize the items to be counted in groups.

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but by applying the same property in a different way, you also have

$$1 + (-1+1) + (-1+1) + (-1+1) + ... = 1 + 0 + 0 + 0 + ... = 1$$

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This algorithm *formalizes* the generic notion of infinite sum by precise rules (or Axioms).

The transfinite sum

The transfinite sum will be denoted in the following way:

$$\sum_{k\in\mathbb{N}} a_k. \tag{1}$$

$\mathbb{N} = \{0, 1, 2, 3, ...\}$ is the set of natural numbers. The notion of a transfinite sum does not coincide with the notion of series;

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$$\sum_{k\in\mathbb{N}} a_k$$

denotes a transfinite sum;

$$\sum_{k=0}^{\infty} a_k$$

denotes a usual series.

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5. (**Comparison rule**) If for m large enough
$$\sum_{k=0}^{m!} a_k \geq \sum_{k=0}^{m!} b_k,$$
then
$$\sum_{k\in\mathbb{N}} a_k \geq \sum_{k\in\mathbb{N}} b_k, .$$

Let's now get acquainted with the idea of transfinite sums. The simplest thing that can come to mind is to add a bit of "1's" and "0's".

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To formalize this fact, it is useful to define the indicator function of $E \subseteq \mathbb{N}$:

$$\chi_{_E}(k) = \begin{cases} 1 & se \quad k \in E \\ 0 & se \quad k \notin E. \end{cases}$$

The numerosity of a subset of natural numbers

For every $E \subset \mathbb{N}$ we can define the number

$$\mathfrak{n}(E) = \sum_{k \in \mathbb{N}} \chi_{_E}(k)$$

which will be called numerosity of E. If E is a finite set, its numerosity corresponds to a natural number.

The numerosity of a subset of natural numbers

For every $E \subset \mathbb{N}$ we can define the number

$$\mathfrak{n}\left(E\right) = \sum_{k \in \mathbb{N}} \chi_{_E}\left(k\right)$$

which will be called numerosity of E. If E is a finite set, its numerosity corresponds to a natural number. Otherwise, the number $\mathfrak{n}\left(E\right)$ is an infinite number that "generalizes" the previous notion.

The number omega

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The ω symbol is the same as it is used to denote the ordinal number relative to the order type of $\mathbb{N}$. This fact is desirable, since the two notions, proceeding in theory, can be identified.

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Notice that

$$ord(\mathbb{N}^+) = ord(\mathbb{N}) = \omega$$

and

$$card(\mathbb{N}^+) = card(\mathbb{N}) = \aleph_0$$

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where $\mathfrak{E} = \{0, 2, 4, 6, 8, ...\}$ is the set of the even numbers .

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where $\mathfrak{Q} = \{1, 4, 9, 16, ...\}$ is the set of the square numbers.

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$$\mathfrak{n}\left(\mathbb{N} \times \mathbb{N}\right) = \omega^2$$

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So the Euclidean continuum has been identified with Dedekind's continuum and the Euclidean straight line has been identified with the set of real numbers (once the origin O and a the unit segment OU have been fixed).
Although this identification is almost universally accepted today, it is still unsatisfactory (not to say wrong) as it contradicts some theorems of Euclidean geometry.

Dedekind's continuum does not model the Euclidean continuum

As an example we consider the following Euclidean statement:

a segment AB can be divided into
two congruent segments AM and MB.

If AB is identified with Dedekind continuum (e.g $[A, B] \subset \mathbb{R}$), then AM has a maximum point or MB has a minimum point.

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Then AM and MB are not congruent, so Dedekind's continuum is not a proper model of the Euclidean continuum.

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Then the image of the Euclidean straight line that comes out is that of a linearly ordered set $\mathfrak{E}$ and the segment AB is a subset of $\mathfrak{E}$ that can not be identified with the set theoretical segment

$$S(A, B) := \{X \in \mathfrak{E} \mid A < X < B\},$$

since

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So we need to represent segment AB as a set of atoms interspersed by empty spaces.

$$|\circledcirc\circledcirc\circledcirc\circledcirc|\circledcirc\circledcirc\circledcirc\circledcirc|\circledcirc\circledcirc\circledcirc\circledcirc|\circledcirc\circledcirc\circledcirc\circledcirc|\circledcirc\circledcirc\circledcirc\circledcirc|\circledcirc\circledcirc\circledcirc\circledcirc|$$

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So we need to represent segment AB as a set of atoms interspersed by empty spaces.

$$|\odot\odot\odot\odot|\odot\odot\odot\odot|\odot\odot\odot\odot|\odot\odot\odot\odot|\odot\odot\odot\odot|\odot\odot\odot\odot|$$

In this model, empty spaces correspond to real numbers.

How to model the Euclidean continuum?

To build a consistent model, we are obliged to assume that points A, B and M **do not belong** to the AB segment.
Then the image of the Euclidean straight line that comes out is that of a linearly ordered set $\mathfrak{E}$ and the segment AB is a subset of $\mathfrak{E}$ that can not be identified with the set theoretical segment

$$S(A, B) := \{X \in \mathfrak{E} \mid A < X < B\},$$

since

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So we need to represent segment AB as a set of atoms interspersed by empty spaces.

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Definition

A set of magnitudes G is said to be Archimedean if given two non-null magnitudes $a, b \in G$, there exists $n \in \mathbb{N}$ such that

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Between 0 and the set of positive numbers, $\mathbb{R}$ has a hole that contradicts our idea of continuum.

Non-Archimedean Geometry and Non-Archimedean Mathematics

Thus, a coherent idea of Euclidean continuum leads us directly to the Non-Archimedean geometry as was conceived by Giuseppe Veronese at the end of the nineteenth century.

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Levi-Civita developed the geometric ideas of Veronese in the direction of the analysis (**Levi-Civita's field**, 1892)

Non-Archimedean Mathematics and Nonstandard Analysis

A more modern approach to Non-Archimedean mathematics is given by the Non-Standard Analysis (ANS) (Robinson 1961) and its variants (e.g. Nelson, 1977, Hrbacek, 2001)

Nonstandard Analysis and Euclidean numbers

The theory of **Euclidean numbers** (which was developed for these needs) is an evolution of the ANS in line with Veronese and Levi-Civita's spirit (B., Forti, 2017).

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Roughly speaking, an Euclidean number is defined as the **transfinite** sum of any arbitrary set of real numbers:

$$\xi = \sum_{k \in E} a_k$$

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The third problem: infinitesimal numbers

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But even the notion of transfinite sums leads us to the notion of infinitesimal.

In fact, by transfinite sums you can get not only infinite but also infinitesimal numbers. Consider, for example, the transfinite sum

$$1-\sum_{k\in\mathbb{N}^+}\frac{1}{2^k}$$

where $\mathbb{N}^+=\{1,2,3,...\}$ is the set of positive natural numbers.

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Strange that Russel took this position, because he also wrote:

Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

B. Russell, *Mysticism and Logic*, 1901.

The existence infinitesimals

Following the ideas of Russell, the word **existence** is synonymous with **consistence**.

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So the correct question is not:

Do infinitesimals exist?

But rather

It is convenient to use infinitesimals.

First application: rounding up the numbers

Consider the number $\frac{1}{3}$. Its decimal form is given by

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But these two quantities are exactly the same?

Rounding a number

If we add the first n terms, we have that

$$\sum_{k=1}^{n} \frac{3}{10^k} = 0,\underbrace{333.....33}_{n \quad \text{digits}} < \frac{1}{3}$$

and therefore for the property 4 of the transfinite sum it follows that

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Conclusion

$$0,33333..... = st\left(\sum_{k \in \mathbb{N}^+} \frac{3}{10^k} \right) := \sum_{k=1}^{\infty} \frac{3}{10^k}$$

The real numbers

This reasoning leads us to a new definition of **real number**.

Definition

A real number is the "**rounding**" (standard part) of a transfinite sum of rational numbers (provided this sum is bounded).

Meaning of the Decimal Representation of Real Number

$$x = a_0, a_1 a_2 a_3 a_4 \ldots.$$

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$$0, 99999\dots\dots = 1$$

The concept of derivative

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The concept of derivative

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Example:

$$\begin{aligned} D(x^2) &= st\left(\frac{(x+\varepsilon)^2 - (x)^2}{\varepsilon}\right) \\ &= st\left(\frac{2x\varepsilon + \varepsilon^2}{\varepsilon}\right) = st\,(2x+\varepsilon) = 2x \end{aligned}$$

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Are infinitesimal necessary?

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So we are lead to talk about problems that can not be treated outside the NAM.

The infinitesimal in empirical sciences

Let us assume the Galilean point of view:

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The infinitesimal in empirical sciences

Let us assume the Galilean point of view:

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and let us see some phenomena that can not be described (easily) without using infinite and infinitesimal numbers.

Calculus of Probability

Limitations of Calculus of Probability based on Kolmogorov Axioms

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Kolmogorov's axioms embed the calculus of probability into the measure theory.

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The sets of null probability represent very rare events. But not impossible. But all this leads to trouble, not only epistemological, but also **technical**.

Technical consequences of all this

Problem

If a meteorite has fallen at the longitude of 11 ° E, what is the probability that it has fallen within a radius of 100 km from Florence.

Our problem is solved by the conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

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In our case,

$$\begin{aligned} A &= \{\text{a meteorite fell within a radius of 100 km from Florence.}\} \\ B &= \{\text{a meteorite fell to the } 11^\circ\text{-E longitude}\} \end{aligned}$$

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$$\frac{0}{0}$$

is a number prohibited by all laws !!!

The non Archimedean Probability (NAP)

A NAP-space is defined by the pair (Ω, w) where Ω is the event space and

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So, the probability of an event A is defined by the following number

$$P(A) = \frac{\sum_{\omega \in A} w(\omega)}{\sum_{\omega \in \Omega} w(\omega)}.$$

This is the trivial definition of probability when Ω is a finite set.

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provided that we accept the transfinite sums
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infinite and infinitesimal numbers

Properties of the non-Archimedean Probability

- (NAP0) **Domain and range.** *The events are the subsets of* Ω *and the probability is a function*

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where $\mathfrak{R}$ *is an ordered field.*

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There are only three small differences with Kolmogorov's axioms. Moreover the "Continuity Axiom" is not here since we have the transfinite sum algorithm

Fair probability

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The probability of an event is given by the ratio between the number of favorable cases $\mathfrak{n}(A)$ and the number of all possible cases $\mathfrak{n}(\Omega)$(the old, dear and tautological classical definition of Laplace).

De Finetti's lottery

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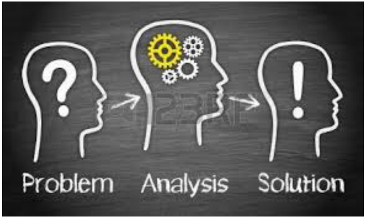
For example, if $A = \{1, 2, 3\}$

$$P(A) = \frac{\mathfrak{n}(A)}{\mathfrak{n}(\mathbb{N})} = \frac{3}{\omega} \sim 0.$$

Exercise 1

What is the probability that De Finetti's lottery comes out is an even number?



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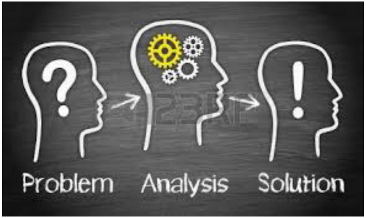

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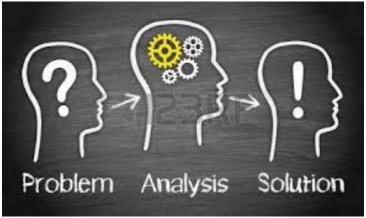

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This is just the example that De Finetti used to criticize Kolmogorovian's probability, unable to model this problem.

Exercise 2

What is the probability that number 1 will come out, knowing from the oracle that one of the first 10 numbers will come out?

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What is the probability that number 1 will come out, knowing from the oracle that one of the first 10 numbers will come out?

The conditional probability $P(\{1\} \mid \{0, 1, 2, 3, ..., 9\})$ is given by

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$\frac{0}{0}$ *is unlawful,*

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$$\frac{0}{0} \text{ is unlawful, but } \frac{\frac{1}{\omega}}{\frac{10}{\omega}} \text{ is allowed.}$$

Some bibliography

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The peculiarity of ultrafunctions is that they are based on a Non-Archimedean field.

Ultrafunctions

Given an set $\Omega \subset \mathbb{R}^N$ the set of ultrafunctions $V^\circ(\Gamma)$ is a $\mathbb{E}$-algebra of functions

$$u : \Gamma \to \mathbb{E}$$

where

$$\Omega \subset \Gamma \subset \mathbb{E}^N$$

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can be extended to an ultrafunction
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- if $f \in C^1$, then for every $x \in \Omega$,
$$f'(x) = Df^{\circ}(x)$$
where
$$D : V^{\circ}(\Gamma) \rightarrow V^{\circ}(\Gamma)$$

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- Every distribution T can be associated to an ultrafunction u_T such that $\forall \varphi \in C^{\infty}_{comp}$
$$\langle T, \varphi\rangle = \sum_{x\in\Gamma} u_T(x)\varphi(x)d(x)$$

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We want to know the point P_0 which the particle will occupy.

Natural way to model the problem

Given a point $P \in \Omega$, consider the Dirichlet problem

$$\begin{cases} -\Delta u = \delta_P & \text{for } \; x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega \end{cases}$$

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Denote by u_P the solution of the above problem and by

$$E_{el}(u_P) = \langle \delta_P , \ u \rangle - \frac{1}{2} \int |\nabla u_P|^2 dx = \frac{1}{2} \int |\nabla u_P|^2 dx$$

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its energy.
The point P_0 is the point which minimizes the energy:

$$\min_{P \in \Omega} E_{el}(u_P)$$

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In fact, for every $P \in \overline{\Omega}$, $E_{el}(u_P)$ is an (infinite) Euclidean number which can be estimated easily. The minimum is achieved by any point $P \in \partial\Omega$.

Some bibliography

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Thank you for your attention

A belief in the infinitely small does not triumph easily. Yet when one thinks boldly and freely, the initial distrust will soon mellow into a pleasant certainty.

Paul du Bois-Reymond