Mean field equations and the global bifurcation diagram of the Gel'fand Problem

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University of Rome "Tor Vergata"

Variational methods with applications to problems in mathematical physics and geometry

On the occasion of the 75th birthday of Prof. Antonio Ambrosetti

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A joint research project in collaboration with:
A. Jevnikar (Univ. Udine, Italy),

- **[B]** D.B., "*Global bifurcation analysis of mean field equations and the Onsager microcanonical description of two-dimensional turbulence*", Calc. Var & P.D.E. **(2019)**.
- **[BJ]** D.B., A. Jevnikar, "*On the global bifurcation diagram of the Gel'fand problem*", arXiv:1901.06700v1.

The Gel'fand Problem

We are concerned with the global bifurcation diagram of solutions of the Gel'fand problem,

$$\begin{cases} -\Delta v = \mu e^{v} \quad \text{in} \quad \Omega \\ \\ \quad v = 0 \qquad \text{on} \quad \partial\Omega \end{cases} \qquad (\mathbf{Q}_{\mu,\Omega})$$

where,

- $\Omega \subset \mathbb{R}^2$ is any open, smooth and bounded domain;
- $\mu \in (-\infty, +\infty)$.

The Gel'fand Problem

The Gel'fand problem $(\mathbf{Q}_{\mu,\Omega})$ arises in many applications, such as for example:

- the nonlinear heat diffusion and thermal ignition of gases [Frank-Kamenetskii **(1955)**, Gel'fand **(1963)**, Bebernes & Eberly **(1989)**];
- the statistical mechanics of point vortices in turbulent flows and plasmas [Caglioti, Lions, Marchioro & Pulvirenti **(1995)**, Smith & O'Neil **(1990)**] and of self-gravitating objects with cylindrical symmetries [Ostriker **(1964)**, Katz & Lynden-Bell **(1978)**].

Similar problems with exponential nonlinearities in 2-d also arise in the theory of Chern-Simons and Gauge Fields [Tarantello **(1996)**, Yang **(2001)**], in Liouville Quantum Gravity [Polyakov **(1981)**] and in the dynamics of bacterial chemotaxis [Suzuki **(2005)**].
Moreover, the equation in $(\mathbf{Q}_{\mu,\Omega})$ (known as the Liouville equation [Liouville **(1853)**]) has a long history in mathematics due to its relevance to the study of constant curvature metrics in 2-d. [Picard **(1893)**, Poincarè **(1898)**, Berger **(1971)**, Kazdan & Warner **(1974)**, S.Y.A. Chang & P.C. Yang **(1987)**].

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The global bifurcation diagram on $\Omega = B_R$

Nevertheless, a kind of basic question seems unanswered so far concerning the shape of the unbounded continuum [Rabinowitz **(1981)**] of solutions of $(\mathbf{Q}_{\mu,\Omega})$,

$$\Gamma_\infty(\Omega) = \left\{ (\mu, v_\mu) \in \mathbb{R} \times C_0^{2,\alpha}(\,\overline{\Omega}\,) : v_\mu \text{ solves } (\mathbf{Q}_{\mu,\Omega}) \right\},$$

emanating from the origin $(\mu, v_\mu) = (0,0) \in \Gamma_\infty(\Omega)$.

Question: Under which conditions on Ω, $\Gamma_\infty(\Omega)$ takes the same form (Fig. [1]) as that corresponding to a disk $\Omega = B_R$?

Here $B_R = \{x \in \mathbb{R}^2 \,:\, |x| < R\}$ and in this case solutions are radial [Gidas, Ni & Nirenberg **(1979)**] and can be evaluated explicitly.

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The global bifurcation diagram on $\Omega = B_1$

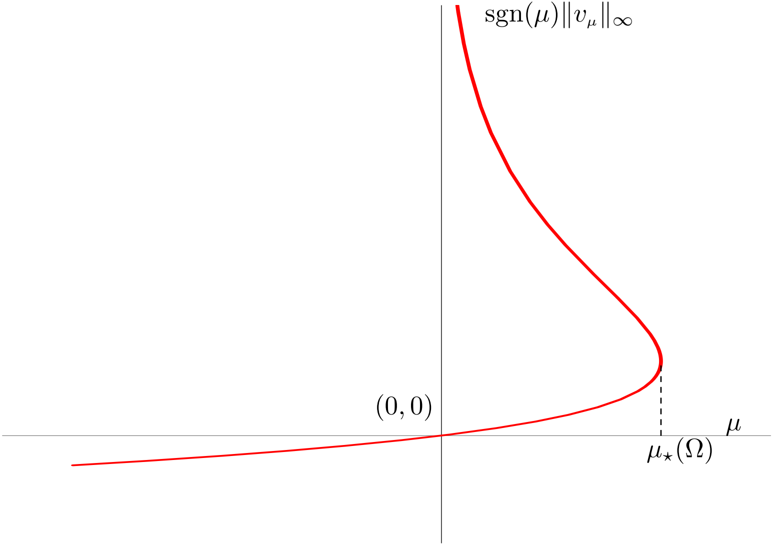

Figure: [1] The graph of $\Gamma_\infty(B_1)$

Well known facts about $\Gamma_\infty(\Omega)$

Let us list some well known facts about $\Gamma_\infty(\Omega)$.

Classical results [Keller & Cohen **(1967)**], [Keener & Keller **(1974)**], [Crandall & Rabinowitz **(1975)**] show that v_μ is a monotonic increasing function of μ as far as the first eigenvalue of the associated linearized problem is strictly positive. This is the so called branch of **minimal solutions** which is well understood and naturally described in the $(\mu, \mathrm{sgn}(\mu)\|v_\mu\|_{L^\infty(\Omega)})$-plane for $\mu < \mu_\star(\Omega) \in (0, +\infty)$.

In fact, this nice behavior breaks down at some positive value $\mu_\star(\Omega)$, which is the least upper bound of those $\mu_\star$ such that $(\mathbf{Q}_{\mu,\Omega})$ has solutions for any $\mu < \mu_\star$. In particular the first eigenvalue of the linearized problem at $\mu_\star(\Omega)$ is zero and $\mu_\star(\Omega)$ is known to be a left bending point.

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The global bifurcation diagram: minimal solutions and the bending point in $\Gamma_\infty(\Omega)$

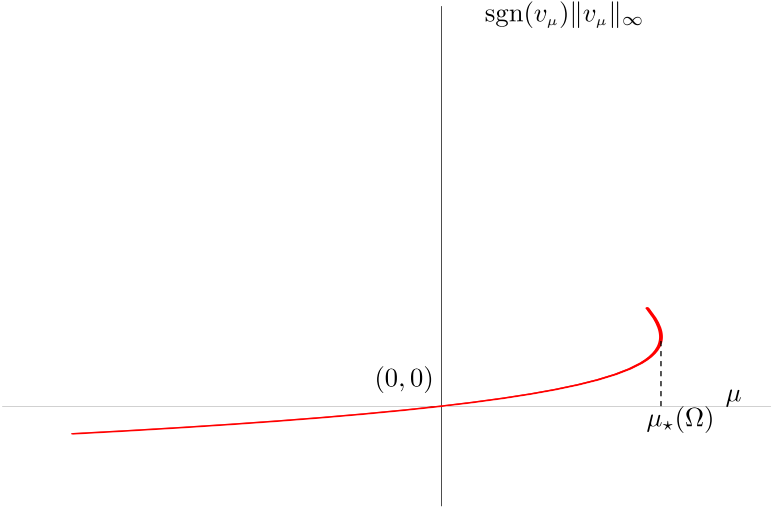

Figure: The branch of minimal solutions

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The situation for non-minimal solutions on $\Gamma_\infty(\Omega)$, i.e. after the first bending point, is more involved.

Based on the Mountain Pass Theorem [Ambrosetti & Rabinowitz **(1973)**], it has been shown in [Crandall & Rabinowitz **(1975)**] that for any $\mu \in (0, \mu_\star(\Omega))$ there exists at least another solution of $(\mathbf{Q}_{\mu,\Omega})$.

On the other side, if Ω has at least one hole, then for any $N \in \mathbb{N}$ there exists a sequence of solutions of $(\mathbf{Q}_{\mu,\Omega})$ making N-point blow up as $\mu_n \to 0^+$, (the concentration phenomenon) that is,

$$\mu_n e^{v_{\mu_n}} \to 8\pi \sum_{i=1}^{N} \delta_{p_i}, \text{ as } n \to +\infty,$$

where $\{p_1, \cdots, p_N\} \subset \Omega$, [Esposito, Grossi & Pistoia **(2005)**, Del Pino, Kowalczyk & Musso **(2005)**].

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Therefore, in general, there can be many solutions of $(\mathbf{Q}_{\mu,\Omega})$ for μ small and "high" $\|v_\mu\|_\infty$, and in particular it may happen in this situation that $\Gamma_\infty(\Omega)$ has many bifurcating branches, see for example [Nagasaki & Suzuki **(1990)**] when Ω is an annulus.

Well known facts about $\Gamma_\infty(\Omega)$

For a particular class of simply connected domains it has been proved in [Suzuki **(1992)**], that $\Gamma_\infty(\Omega)$ is a smooth curve with only one bending point, which makes 1-point blow up as $\mu \to 0^+$ (i.e. $\mu e^{v_\mu} \to 8\pi\delta_p$, $p \in \Omega$, as $\mu \to 0^+$).

As a consequence of a result in [D.B. & C.S. Lin **(2014)**], this is true also for a particular class of non simply connected domains.

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An open problem about the monotonicity of non minimal solutions in $\Gamma_\infty(\Omega)$

On the other side, at this point a subtle problem arise. Indeed, for $\mu > 0$, we have that

$$\lambda := \mu \int\limits_{\Omega} e^{v_\mu}, \text{ takes values in } (0, 8\pi), \text{ along } \Gamma_\infty(\Omega),$$

while **the first eigenvalue ν_1 of the linearized equation for $(\mathbf{Q}_{\mu,\Omega})$ is positive if $\lambda < 4\pi$ but then changes sign at some $\lambda_* \geq 4\pi$ and stays negative in $(\lambda_*, 8\pi)$ (non minimal branch). The second eigenvalue ν_2 is always positive for $\lambda < 8\pi$**, see [Suzuki **(1992)**] for simply connected domains and [D.B. & C.S. Lin **(2014)**] for general domains.

Therefore, after the bending, **which happens at some $\lambda_* \geq 4\pi$, we have $\nu_1 < 0$**, and then, even in this lucky situation, where we know that the bifurcation curve cannot bend back to the right, we don't know much about the monotonicity of $\|v_\mu\|_\infty = \|v_\mu\|_{L^\infty(\Omega)}$.

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Possible non monotone behaviour of the non minimal branch in $\Gamma_\infty(\Omega)$

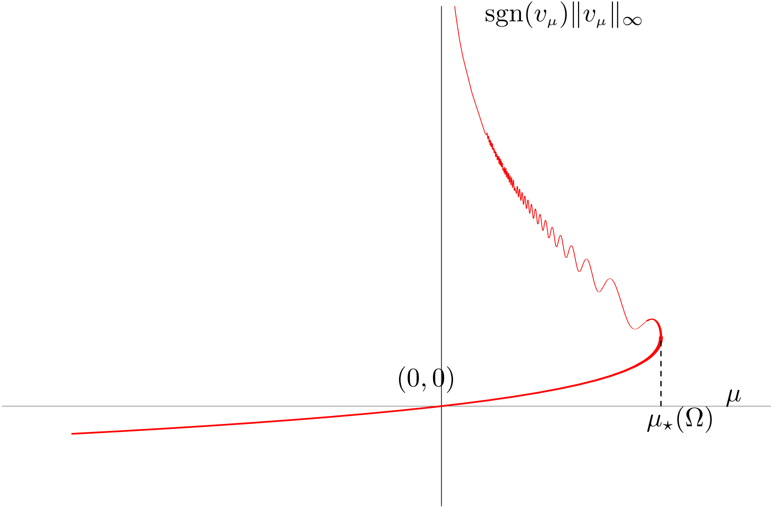

Figure: Possible behaviour of non minimal solutions

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The problem is not trivial because along the minimal branch ν_1 is strictly positive and then (by the maximum principle) $\|v_\mu\|_\infty$ is strictly increasing as μ increases in $(-\infty, \mu_\star(\Omega))$, while, along the non minimal branch, one wants to prove that $\|v_\mu\|_\infty$ **is decreasing as** μ **increases in** $(0, \mu_\star(\Omega))$**, by using only** $\nu_1 < 0$ **and** $\nu_2 > 0$.
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$\Gamma_\infty(\Omega)$ on symmetric and directionally convex domains

Actually, for domains which are symmetric and convex in the coordinates directions, by the result in [Holzmann & Kielhöfer **(1994)**], for any $m \in (0, +\infty)$ there exists one and only one solution of $(\mathbf{Q}_{\mu,\Omega})$ such that $\mathrm{sgn}(\mu)\|v_\mu\|_\infty = m$ and $\Gamma_\infty(\Omega)$ is a smooth curve which contains all solutions of $(\mathbf{Q}_{\mu,\Omega})$.
Therefore, for these domains, $\|v_\mu\|_\infty$ is monotone along $\Gamma_\infty(\Omega)$, which answer to our question in this case.

We attack this problem by a new approach suggested by some recent results in **[B]**, that is, to go back to one of the physical motivations and find out a natural quantity suitable to be used as a **global variable to parametrize** $\Gamma_\infty(\Omega)$.

Indeed, we will not look at the trace of $\Gamma_\infty(\Omega)$ on the $(\mu, \mathrm{sgn}(\mu)\|v_\mu\|_\infty)$ plane, but rather on the $(\mu, \mathbb{E}(\mu))$ plane, where $\mathbb{E}(\mu)$ is the energy of the associated mean field equation, naturally arising in the Onsager statistical mechanics description [Caglioti, Lions, Marchioro & Pulvirenti **(1995)**] of two-dimensional turbulence.

A new approach to the analysis of $\Gamma_\infty(\Omega)$

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For any pair $(\mu, v_\mu) \in (\mathbb{R}, C_0^{2,\alpha}(\overline{\Omega}))$ solving $(\mathbf{Q}_{\mu,\Omega})$ we define,

$$\mathbb{E}(\mu) = \begin{cases} \dfrac{1}{2\mu \int\limits_\Omega e^{v_\mu}} \displaystyle\int\limits_\Omega \dfrac{e^{v_\mu}}{\int\limits_\Omega e^{v_\mu}} v_\mu, & \mu \neq 0, \\ \frac{1}{2|\Omega|^2} \int\limits_\Omega \int\limits_\Omega G(x,y) dx dy, & \mu = 0, \end{cases}$$

where $G(x,y)$ is the Green function for $-\Delta$ with Dirichlet boundary conditions. For later use let us set,

$$E_0 := \mathbb{E}(0) = \frac{1}{2|\Omega|^2} \int\limits_\Omega \int\limits_\Omega G(x,y) dx dy.$$

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The mean field equation

Although its definition in the context of the Gel'fand problem looks rather unnatural, it turns out that indeed $\mathbb{E}(\mu)$ is just the energy in the [Onsager **(1949)**] mean field model of two dimensional turbulence, when expressed as a function of μ. For $\lambda \in (-\infty, +\infty)$ we consider the **mean field equation**,

$$\begin{cases} -\Delta\psi_\lambda = \dfrac{e^{\lambda\psi_\lambda}}{\int\limits_\Omega e^{\lambda\psi_\lambda}} & \Omega \\ \psi_\lambda = 0 & \partial\Omega \end{cases} \qquad \mathbf{(P_{\lambda,\Omega})}$$

This is a particular case of a stationary Euler equation in vorticity form, where ψ_λ is the **stream function** of the flow ($\overrightarrow{\mathrm{v}} = \nabla^\perp\psi_\lambda = (-\partial_2\psi_\lambda, \partial_1\psi_\lambda)$) and

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The kinetic energy associated to the flow is $\frac{1}{2} \int\limits_\Omega |\nabla \psi_\lambda|^2 = \frac{1}{2} \int\limits_\Omega |\overrightarrow{v}|^2$.
Since $\psi_\lambda(x) = \int\limits_\Omega G(x,y) \rho_\lambda(y) dy$, then, integrating by parts in $(\mathbf{P}_{\lambda,\Omega})$, we see that,

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and $v_\mu(x) = \lambda \psi_\lambda(x)$ we have a pair (μ, v_μ) solving $(\mathbf{Q}_{\mu,\Omega})$.

We will see that, for a large class of domains, for $\lambda \in (-\infty, \lambda_*]$ and $[\lambda_*, 8\pi)$ is well defined the inverse of μ_λ, $\lambda = \lambda_\mu$ which allows us to write $\mathbb{E}(\mu) = \frac{1}{2\lambda_\mu} < v_\mu >_{\lambda_\mu} = \frac{1}{2} < \psi_\lambda >_\lambda \big|_{\lambda=\lambda_\mu}$, that is $\mathbb{E}(\mu)$ **is just** $\mathcal{E}(\boldsymbol{\rho}_\lambda)$ when expressed in terms of (μ, v_μ).

The underlying idea is to describe $\Gamma_\infty(\Omega)$ in terms of the solutions of $(\mathbf{P}_{\lambda,\Omega})$ with the energy as a **global parameter**.

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The Microcanonical Variational Principle.

The reason why there is a chance to succeed in using the energy as a global parameter, is that it has been shown in [Caglioti, Lions, Marchioro & Pulvirenti **(1995)**] that for any $E \in (0, +\infty)$, there exists $\lambda = \lambda(E)$ and a solution ψ_λ of $(\mathbf{P}_{\lambda,\Omega})$, which is the stream function of a density ρ which maximizes the entropy

$$\mathfrak{S}(\rho) = -\int_\Omega \rho \log(\rho), \quad \rho \geq 1 \text{ a.e. in } \Omega,$$

for fixed energy $\mathcal{E}(\rho) = E$ and total vorticity $\int_\Omega \rho = 1$. This is the so called microcanonical variational principle. In particular $\lambda(E)$ is the Lagrange multiplier associated to the energy constraint, $\lambda(E) = -\frac{1}{\kappa T_{\rm stat}(E)}$, where $T_{\rm stat}$ is the statistical temperature. Unlike ordinary states, in a negative temperature state vortices attract each other, which is the origin of the concentration phenomenon.

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Existence of solutions of $(\mathbf{P}_{\lambda,\Omega})$ for $\lambda < 8\pi$.

The existence and uniqueness of solutions of $(\mathbf{P}_{\lambda,\Omega})$ is easy to prove for $\lambda \leq 0$. On the other side it is well known that $(\mathbf{P}_{\lambda,\Omega})$ admits a unique solution for any $\lambda \in (0, 8\pi)$ ([Suzuki **(1992)**] for simply connected domains, [D.B., C.S. Lin **(2014)**] for general domains).

The existence/non existence problem for $\lambda = 8\pi$ is a subtle issue, since $(\mathbf{P}_{\lambda,\Omega})$ is critical with respect to the Moser-Trudinger inequality [Moser **(1971)**]. The variational functional whose critical points are solutions of $(\mathbf{P}_{\lambda,\Omega})$ is bounded from below but it is not coercive for $\lambda = 8\pi$. Therefore minimizing sequences are not always convergent. For a complete discussion of this problem see [S.Y.A. Chang, C.C. Chen & C.S. Lin **(2003)**] for simply connected domains and [D.B. & C.S. Lin **(2014)**] for general domains.

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Domains of first/second kind

This is why we need the following definition,

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A domain Ω is of **first kind** if $(\mathbf{P}_{\lambda,\Omega})$ has no solution for $\lambda = 8\pi$ and is of second kind otherwise.

This definition was first introduced with a different (but equivalent) formulation in (Caglioti, Lions, Marchioro & Pulvirenti **(1992)**). It has been proved in [S.Y.A. Chang, C.C. Chen & C.S. Lin **(2003)**] for simply connected domains and in [D.B. & C.S. Lin **(2014)**] for general domains that minimizing sequences converge, if and only if Ω is of second kind and that Ω is of first kind if and only if the unique solutions of $(\mathbf{P}_{\lambda,\Omega})$ for $\lambda < 8\pi$ make 1-point blow up as $\lambda \to 8\pi^-$. As a corollary we have a characterization of the simply connected domains considered in [Suzuki **(1992)**]: it turns out a posteriori that they are exactly the simply connected domains of first kind.

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If $\Omega_{a,b}$ is a rectangle of sides $a \leq b$, then there exists $\xi \in (0,1)$ such that $\Omega_{a,b}$ is of second kind if and only if $\frac{a}{b} < \xi$, see [S.Y.A. Chang, C.C. Chen & C.S. Lin **(2003)**]. If $\Omega_r = B_1 \setminus \overline{B_r(x_0)}$ with $x_0 \in B_1$, $x_0 \neq 0$, then there exists $r_0 < \min\{|x_0|, 1-|x_0|\}$ such that Ω_r is of first kind for any $r < r_0$, see [D.B. & C.S. Lin **(2014)**].

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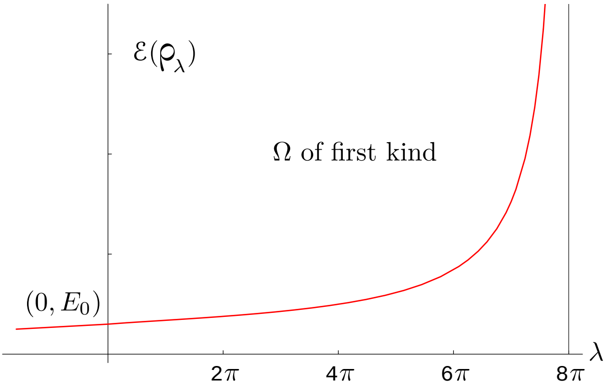
We say that $f : I \to X$, where $I \subseteq \mathbb{R}$ is an open set and X is a Banach space, is real analytic if for each $t_0 \in I$ it admits a power series expansion in t, which is totally convergent in the X-norm in a suitable neighborhood of t_0.

The bifurcation diagram for $\lambda < 8\pi$ on a domain of first kind

Our first result **[B,BJ]** is relevant also to the mean field equation on domains of first kind and asserts that the energy $\mathcal{E}(\rho_\lambda)$ is a real analytic and strictly increasing for $\lambda \in (-\infty, 8\pi)$.



Then we have,

Theorem 1 [BJ]

Let Ω be a domain of **first kind**. For any $E \in (0,+\infty)$, the equation

$$\mathbb{E}(\mu) = E \qquad (\mu, v_{\mu}) \in \Gamma_\infty(\Omega) \qquad \textbf{(E)}$$

admits a unique solution $\mu = \mu_\infty(E)$.
In particular, $\mu_\infty : (0,+\infty) \to (-\infty, \mu_\star(\Omega)]$ and $v_{\mu}\big|_{\mu=\mu_\infty(E)} : (0,+\infty) \to C^{2,\alpha}_0(\overline{\Omega})$ are real analytic functions of E and $(\mu, v_{\mu})\big|_{\mu=\mu_\infty(E)}$ is a parametrization of $\Gamma_\infty(\Omega)$.
Finally $\mu_\infty(E)$ has the following properties:
(i) $\mu_\infty(E) \to -\infty$ as $E \to 0^+$, $\mu_\infty(E_0) = 0$, $\mu_\infty(E) \to 0^+$ as $E \to +\infty$;
(ii) $\frac{d\mu_\infty(E)}{dE} > 0$ for $E < E_*$, $\frac{d\mu_\infty(E_*)}{dE} = 0$, $\frac{d\mu_\infty(E)}{dE} < 0$ for $E > E_*$,
where $E_* = E_*(\Omega) > E_0(\Omega)$ is uniquely defined by $E_*(\Omega) = \mathbb{E}(\mu_\star(\Omega))$,
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The shape of $\Gamma_\infty(\Omega)$ on domains of first kind

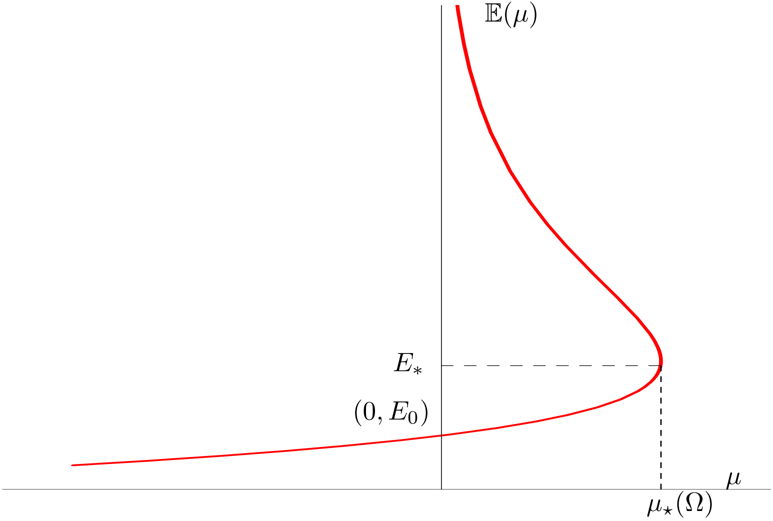

Figure: [4] The graph of $\Gamma_\infty(\Omega)$ on domains of first kind

Therefore, on domains of first kind, we have found a global parametrization of $\Gamma_\infty(\Omega)$,

$$\Gamma_\infty(\Omega) = \left\{(\mu, v_\mu) \in [0, \mu_\star(\Omega)] \times C_0^{2,\alpha}(\,\overline{\Omega}\,) \,:\, \mu = \mu_\infty(E),\; E \in (0,+\infty)\right\},$$

which takes the form depicted in Fig. [4], as claimed. At least to our knowledge, this is the first global result (i.e. including non-minimal solutions) about the shape of the Rabinowitz unbounded continuum, for an elliptic equation with superlinear growth in dimension $n = 2$, which is not just concerned with radially symmetric solutions [Korman **(2012)**], and/or with domains symmetric and directionally convex w.r.t. two orthogonal directions [Holzmann & Kielhöfer **(1994)**].

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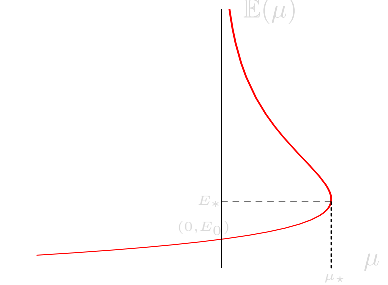
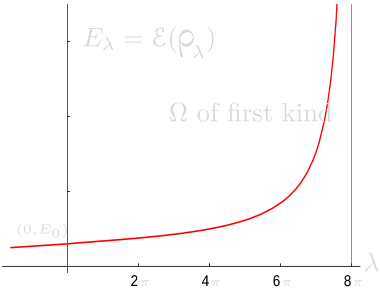
The underlying idea is to parametrize $\Gamma_\infty(\Omega)$ via the map $\lambda \mapsto \mu_\lambda$ along the analytic branch $\mathcal{G}_{8\pi}$ of solutions of $(\mathbf{P}_{\lambda,\Omega})$ for $\lambda \in (-\infty, 8\pi)$.
As a matter of fact, the natural **global parameter** (at least for $\lambda < 8\pi$) to describe both $(\mathbf{Q}_{\mu,\Omega})$ and $(\mathbf{P}_{\lambda,\Omega})$, is the energy.





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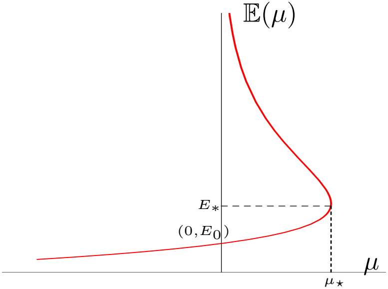

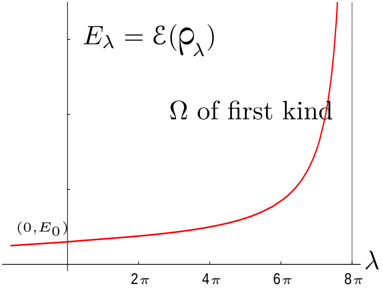

Sketch of the proof

The use of $(\mathbf{P}_{\lambda,\Omega})$ as an auxiliary problem is very helpful since, unlike $(\mathbf{Q}_{\mu,\Omega})$ where the Morse index along the non minimal branch is 1, the **Morse index of the solutions of $(\mathbf{P}_{\lambda,\Omega})$ is zero along** $\mathcal{G}_{8\pi}$, as shown in [Suzuki **(1992)**, D.B. & C.S. Lin **(2014)**].

On the other side it seems that we have a high price to pay, since $(\mathbf{P}_{\lambda,\Omega})$ is a constrained problem and then the eigenvalue problem associated to the linearized equation

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From the physical point of view ([**B**]), the natural spectral formulation of $(\mathbf{P}_{\lambda,\Omega})$ seems to be the following,

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which corresponds to take variations along the tangent space to the constraint $\int\limits_\Omega \rho_\lambda = 1$. However, in general the first eigenvalue $\sigma_{1,\lambda}$ need not be simple, the first eigenfunctions may change sign and, even if $\sigma_{1,\lambda} > 0$, the maximum principle in general does not hold for the linearized operator, see either [D.B. **(2012)**] or [**BJ**] for an example of this sort.

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Moreover, denoting by $\phi_{k,\lambda}$ the eigenfunctions relative to the eigenvalue $\sigma_{k,\lambda}$, then $\{(\phi_{k,\lambda})_0\}_{k\in\mathbb{N}} \equiv \{\phi_{k,\lambda} - <\phi_{k,\lambda}>_\lambda\}_{k\in\mathbb{N}}$ is a complete base in the subspace of $L^2(\Omega)$ functions of vanishing mean.

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In particular, if we define the Fourier coefficients of $(\psi_\lambda)_0$ and of $(\frac{d\psi_\lambda}{d\lambda})_0$

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As a consequence of $(\mathbf{S1}) - (\mathbf{S2})$ we can prove that $\mathcal{E}(\rho_\lambda)$ is real analytic and strictly increasing, so that **it is just enough to understand the sign of** $\frac{d\mu_\lambda}{d\lambda}$.

Then we have:

(**S3**) An integral version of the maximum principle for $z_\lambda = \frac{d(\lambda\psi_\lambda)}{d\lambda}$ with $\lambda < 8\pi$.

(**S4**) We can prove a subtle property of a certain quantity proportional to $\frac{d\mu_\lambda}{d\lambda}$ **which is seen to satisfy a linear non-homogeneous ODE**. This is the result of a rather surprising evaluation which, together with (**S3**), allows us to conclude that $\frac{d\mu_\lambda}{d\lambda}$ must stay negative along the non minimal branch.

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It is easy to see that,

$$\left(\int\limits_{\Omega} e^{\lambda\psi_\lambda}\right)\frac{d\mu_\lambda}{d\lambda} = 1-\lambda < z_\lambda >_\lambda =: g(\lambda), \lambda \in (-\infty, 8\pi),$$

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To have a grasp of the problem one can observe that z_λ satisfies,

$$\int_\Omega |\nabla z_\lambda|^2 - \lambda < z_\lambda^2 >_\lambda = < z_\lambda >_\lambda (1 - \lambda < z_\lambda >_\lambda) = < z_\lambda >_\lambda g(\lambda).$$

The l.h.s. is just the numerator of the Rayleigh quotient of the **unconstrained problem**, which is known to be strictly positive for $\lambda < 4\pi$ ([Suzuki **(1992)**, D.B. & C.S. Lin **(2014)**]), while for $\lambda \in (\lambda_*, 8\pi)$ **we need to prove that it is strictly negative although only the first eigenvalue is negative.**
In other words, if we would go back and take the Fourier expansion in the unconstrained setting of $g(\lambda)$, then we would find only one negative term against infinitely many positive terms, and we would have to prove that the negative term prevails.

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This is also why it is still non trivial, even in the constrained setting, to prevent the oscillations of $g(\lambda) = 1 - \lambda < z_\lambda >_\lambda$, that is, to show that indeed it changes sign **only once** in $\lambda \in (0, 8\pi)$. This is obtained as a consequence of $(\mathbf{S3}) - (\mathbf{S4})$.

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In particular, concerning (**S4**), we have found a surprising property of $g(\lambda)$, which is seen to satisfy the following first order non-homogeneous linear O.D.E.:

$$g^{'}(\lambda) = a(\lambda)g(\lambda) + b(\lambda), \quad \lambda \in (-\infty, 8\pi), \qquad g(0) = 1,$$

where

$$a(\lambda) = -(2\lambda < (z_\lambda)_0^2 > +\lambda < z_\lambda^2 > + < z_\lambda >), \quad b(\lambda) = -\lambda^2 < z_\lambda^3 > .$$

By a careful analysis of this equation, we conclude that if $\lambda_* \geq 4\pi$ is the "first" zero of $g(\lambda)$, then $g(\lambda)$ is strictly positive for $\lambda < \lambda_*$ and strictly negative for $\lambda > \lambda_*$.

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Thank you very much for your attention!

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Since the first eigenvalue $\sigma_{1,\lambda}$ is strictly positive for $\lambda \in (-\infty, 8\pi)$, then $\mathcal{G}_{8\pi}$ is a real analytic curve with **no bending and no bifurcation points**.

A crucial fact which follows from (**S1**), (**S2**) and $\sigma_{1,\lambda} > 0$, is that the energy $E_\lambda = \mathcal{E}(\rho_\lambda)$ is a (real analytic) strictly increasing function of λ along $\mathcal{G}_{8\pi}$.

Proposition [B],[BJ]

$$\frac{dE_\lambda}{d\lambda} \geq \frac{\lambda + \sigma_{1,\lambda}}{\sigma_{1,\lambda}} < (\psi_\lambda)_0^2 >_\lambda > 0, \; \forall \lambda \in (-\infty, 0),$$

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Please observe that this is always true (i.e. **also on domains of second kind** ([B])). Therefore, the energy is always a good global variable to describe $\mathcal{G}_{8\pi}$. Moreover, **to understand the monotonicity of μ as a function of E, it is enough to evaluate the sign of $\frac{d\mu_\lambda}{d\lambda}$**.

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First of all, still by $\sigma_{1,\lambda} > 0$, we obtain a version of the maximum principle for z_λ based on the sign of $g(\lambda)$. This is not at all obvious since $\sigma_{1,\lambda} > 0$ **does not imply, as usually happens in unconstrained-type problems, that the maximum principle holds for the linearized problem relative to $(\mathrm{P}_{\lambda,\Omega})$.**
This is readily seen by observing that any eigenfunction (therefore in particular the first eigenfunction) satisfies

$$-\int\limits_{\partial\Omega} \partial_\nu \phi_{k,\lambda} = -\int\limits_{\Omega} \Delta \phi_{k,\lambda} = (\lambda + \sigma_{k,\lambda}) \int\limits_{\Omega} \rho_\lambda (\phi_{k,\lambda} - < \phi_{k,\lambda} >_\lambda) = 0,$$

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Nevertheless we succeed in showing that:

Proposition [BJ]

(i) If $\lambda \in (-\infty, 8\pi)$, then $< z_\lambda >_\lambda > 0$.
(ii) If $\lambda \in (-\infty, 4\pi)$, then $g(\lambda) = 1 - \lambda < z_\lambda >> 0$.
(iii) If $\lambda \in (-\infty, 8\pi)$ and $g(\lambda) \geq 0$, then $z_\lambda(x) \geq 0$ in Ω.

(i) is an immediate consequence of $\sigma_{1,\lambda} > 0$ and of the equation satisfied by z_λ. (ii) follows from (i) and the fact that,

$$0 < \nu_{1,\lambda} < z_\lambda^2 >_\lambda \leq \int\limits_\Omega |\nabla z_\lambda|^2 - \lambda < z_\lambda^2 >_\lambda = < z_\lambda >_\lambda (1 - \lambda < z_\lambda >_\lambda).$$

Here $\nu_{1,\lambda}$ is the first eigenvalue of the **unconstrained problem**, which is known to be positive for $\lambda < 4\pi$ ([Suzuki **(1992)**, D.B. & C.S. Lin **(2014)**]), which yields the l.h.s. inequality. (iii) is obtained by a careful non-local refinement of the usual argument (for unconstrained problems) which shows that the maximum principle holds if the first eigenvalue is positive.

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However, it turns out that this is not enough. To have a grasp of the problem one can observe that z_λ satisfies,

$$\int_\Omega |\nabla z_\lambda|^2 - \lambda < z_\lambda^2 >_\lambda = < z_\lambda >_\lambda (1 - \lambda < z_\lambda >_\lambda).$$

Here we see that the sign of the l.h.s. depends on the eigenvalues of the **unconstrained problem**, which for $\lambda \in (\lambda_*, 8\pi)$ has a **negative first eigenvalue and positive second eigenvalue**. Therefore it is non trivial to prevent the oscillations of this quantity, i.e.,to show that $g(\lambda) = 1 - \lambda < z_\lambda >_\lambda$, which must change sign at least once (since $< z_\lambda >_\lambda \to +\infty$ as $\lambda \to 8\pi^-$), indeed changes sign **only once** in $\lambda \in (0, 8\pi)$.

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We solve this problem by the discovery of a surprising property of $g(\lambda)$, which is seen to satisfy the following first order non-homogeneous linear O.D.E.:

$$g^{'}(\lambda) = a(\lambda)g(\lambda) + b(\lambda), \quad \lambda \in (-\infty, 8\pi), \qquad g(0) = 1,$$

where

$$a(\lambda) = -(2\lambda < (z_\lambda)_0^2 > + \lambda < z_\lambda^2 > + < z_\lambda >), \quad b(\lambda) = -\lambda^2 < z_\lambda^3 > .$$

By a careful analysis of this equation with (iii) above, we conclude that if $\lambda_* \geq 4\pi$ is the "first" zero of $g(\lambda)$, then $g(\lambda)$ is strictly positive for $\lambda < \lambda_*$ and strictly negative for $\lambda > \lambda_*$.

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Appendix: $\Omega = B_1$

For a disk we have the explicit expressions of $\mu = \mu_\lambda, v_\mu = \lambda\psi_\lambda$. Setting

$$\gamma_\lambda = \frac{\lambda}{8\pi - \lambda}, \quad \lambda \in (-\infty, 8\pi),$$

we have

$$\mu_\lambda = \frac{8\gamma_\lambda}{(1+\gamma_\lambda)^2} = \frac{1}{8\pi^2}\lambda(8\pi - \lambda),$$

$$v_\mu = 2\log\left(\frac{1+\gamma_\lambda^2}{1+\gamma_\lambda^2|x|^2}\right), \quad |x| \leq 1.$$

In particular

$$E_\lambda = -\frac{1}{\lambda}\left(1 + \frac{8\pi}{\lambda}\log\left(1 - \frac{\lambda}{8\pi}\right)\right), \qquad E_0 = \frac{1}{16\pi},$$

and $\lambda_*(B_1) = 4\pi$, $E_*(B_1) = \frac{2\log(2)-1}{4\pi}$ and $\mu_\star(B_1) = 2$.

Appendix: non simple first eigenvalue for $(\mathbf{L}_\lambda)$-type problems.

If $\widetilde{\sigma}_1$ is the standard first eigenvalue,

$$-\Delta\phi - V\phi = \widetilde{\sigma}_1\phi_1 \quad \text{in} \quad \Omega$$

then its eigenfunction ϕ_1 is of fixed sign, since the quadratic form in the Rayleigh quotient,

$$\int\limits_{\Omega} \left(|\nabla\phi|^2 - V\phi^2\right)$$

is invariant in $H_0^1(\Omega)$ under the map $\phi \mapsto |\phi|$.

Appendix: non simple first eigenvalue for $(\mathbf{L}_\lambda)$-type problems.

For constrained type problems of the form,

$$-\Delta\phi - V(\phi - <\phi>) = \sigma_1 V(\phi - <\phi>) \quad \text{in} \quad \Omega,$$

with $V > 0$ in Ω and $<\phi> = (\int\limits_{\Omega} V)^{-1} \int\limits_{\Omega} V\phi$, the same argument fails, since the quadratic form now reads,

$$\int\limits_{\Omega} \left(|\nabla\phi|^2 - V(\phi^2 - <\phi>^2)\right),$$

which in general **is not** invariant under the the map $\phi \mapsto |\phi|$, unless $<\phi>$ vanishes, which already implies that ϕ changes sign in Ω.

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Appendix: non simple first eigenvalue for $(\mathbf{L}_\lambda)$-type problems.

We consider a simplified linear problem which however share the same structure of $(\mathbf{L}_\lambda)$.

$$-\Delta\phi = \sigma\left(\phi - \fint_{B_1}\phi\right) \quad \text{in} \quad B_1 \tag{1}$$

with $\phi = 0$ on ∂B_1, and $\fint_{B_1}\phi = \frac{1}{|B_1|}\int_{B_1}\phi$. Passing to the new variable $\phi_0 = \phi - \fint_{B_1}\phi$ we are reduced to calculate the spectrum of

$$-\Delta\phi_0 - \sigma\phi_0 = 0 \quad \text{in} \quad B_1, \tag{2}$$

with boundary conditions

$$\text{(I)} \quad \phi_0 = \text{constant} \quad \text{on} \quad \partial B_1 \quad \text{and} \quad \text{(II)} \quad \fint_{B_1}\phi_0 = 0. \tag{3}$$

Appendix: non simple first eigenvalue for $(\mathbf{L}_\lambda)$-type problems.

Passing to polar coordinates, we first consider solutions of (2) of the form

$$\psi_n(r,\theta) = (A\cos(n\theta) + B\sin(n\theta))\, J_n(\sqrt{\sigma} r)$$

where J_n is the Bessel function of order n. Now either $n = 0$ and then (3)-(I) is always satisfied or $n \geq 1$ and then (3)-(I) is satisfied if and only if $\sigma = \sigma_{n,m} = \mu^2_{n,m}$, where $\mu_{n,m}$ is the m-th zero of J_n. Next, if $n \geq 1$ then (3)-(II) is always satisfied, while if $n = 0$ then (3)-(II) is equivalent to $\int\limits_0^1 J_0(\sqrt{\sigma} r) r dr = 0$. Since $(J_1(r)r)^{'} = rJ_0(r)$ this is equivalent to $\sqrt{\sigma} J_1(\sqrt{\sigma}) = 0$, that is $\sigma = \mu^2_{1,m}$. Therefore the eigenvalues of (1) are

$$\sigma_{n,m} = \mu^2_{n,m},$$

where $\sigma_{1,m}$ admits three eigenfunctions, $\{J_0(\mu_{1,m} r) - J_0(\mu_{1,m}), \cos(\theta) J_1(\mu_{1,m} r), \sin(\theta) J_1(\mu_{1,m} r)\}$ and $\sigma_{n,m}$ with $n \geq 2$ that admits two eigenfunctions $\{\cos(n\theta) J_n(\mu_{n,m} r), \sin(\theta) J_n(\mu_{n,m} r)\})$. Observe that these are the eigenfunctions of (1).

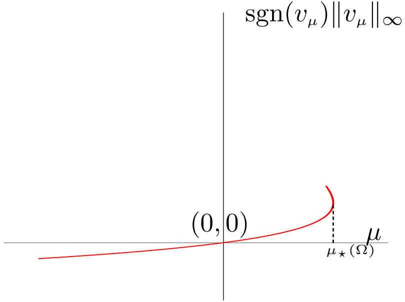
Appendix: non simple first eigenvalue for $(\mathbf{L}_\lambda)$-type problems.

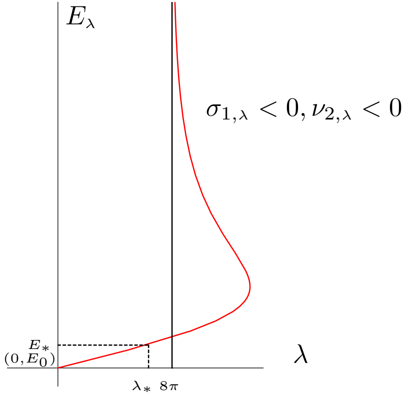
In particular the first eigenvalue $\sigma_{1,1} = \mu_{1,1}^2 \simeq (3.83)^2$ admits three eigenfunctions, one of which is radial,

$$\phi_1 = \phi_1(r) = J_0(\mu_{1,1} r) - J_0(\mu_{1,1}),$$

and satisfies $\phi_1(1) = 0$ and $\phi_1'(1) = \mu_{1,1} J_0'(\mu_{1,1}) = 0$. This is not in contradiction with the Hopf Lemma, since $\displaystyle\fint_{B_1} \phi_1 = -J_0(\mu_{1,1}) \neq 0$ and the identically zero function $\phi \equiv 0$ is not a solution of (1) with $\displaystyle\fint_{B_1} \phi \neq 0$. Alternatively, in terms of ϕ_0, the eigenfunction $\displaystyle\phi_{1,0} = \phi_1 - \fint_{B_1} \phi_1$ satisfies $\phi_{1,0}(1) = J_0(\mu_{1,1}) \neq 0$ and $\phi_{1,0}'(1) = \mu_{1,1} J_0'(\mu_{1,1}) = 0$ but still the function $\phi_0 \equiv J_0(\mu_{1,1}) \neq 0$ is not a solution of (2), (3).

Appendix: domains of second kind with no bifurcation.





The reason why Ω is of first kind if and only if the unique solutions of $(\mathbf{P}_{\lambda,\Omega})$ blow up as $\lambda \to 8\pi^-$ is that, see [S.Y.A. Chang, C.C. Chen & C.S. Lin **(2003)**] and [D.B. & C.S. Lin **(2014)**], if a solution for $\lambda = 8\pi$ exists, then $\widehat{\sigma}_{1,8\pi} > 0$. As a consequence, by the implicit function theorem, we see that there exists a smooth branch of solutions in a full neighborhood of $\lambda = 8\pi$. Therefore, if this branch of bounded solutions exists, then by the uniqueness for $\lambda < 8\pi$, solutions blowing up as $\lambda \to 8\pi^-$ are not allowed. This argument shows that if the unique solutions of $(\mathbf{P}_{\lambda,\Omega})$ blow up as $\lambda \to 8\pi^-$, then there is no solution for $\lambda = 8\pi$, that is, Ω is of first kind. The other implication is easier.

Let,

$$J_\lambda(u) = \frac{1}{2}\int\limits_{\Omega} |\nabla u|^2 - \lambda \log\left(\int\limits_{\Omega} e^u\right), \quad u \in H_0^1(\Omega),$$

$R(x,y) = G(x,y) + \frac{1}{2\pi}\log(|x-y|)$ and $\gamma(x) = R(x,x)$ be the Robin function of Ω. Let us recall that $\gamma(x) \to -\infty$ as $x \to \partial\Omega$.
Then, see [S.Y.A. Chang, C.C. Chen & C.S. Lin **(2003)**] and [D.B. & C.S. Lin **(2014)**],

$$\frac{1}{8\pi}\inf_{H_0^1(\Omega)} J_{8\pi} \leq -1 - \log(\pi) - 4\pi \sup_{x\in\Omega}\gamma(x),$$

and Ω is of second kind if and only if the strict inequality holds.

Appendix: domains of first/second kind.

Also, let $q \in \Omega$ be a critical point of γ and

$$\pi A_\Omega(q) = \lim_{\varepsilon \to 0} \int\limits_{\Omega \setminus B_\varepsilon(q)} \frac{e^{8\pi(R(x,q)-R(q,q))} - 1}{|x-q|^4} - \int\limits_{\mathbb{R}^2 \setminus \Omega} \frac{1}{|x-q|^4}.$$

If Ω is simply connected and $f : B_1 \mapsto \Omega$ is a Riemann map satisfying,

$$f(z) = q + \sum_{n=1}^{+\infty} a_n z^n, \quad z \in B_1 \subset \mathbb{C},$$

then since q is a critical point of γ, we have $a_2 = 0$ and $D_\Omega(q) := |a_1|^2 A_\Omega(q)$ takes the form,

$$D_\Omega(q) = \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n|^2 - |a_1|^2.$$

Observe that $D_{\Omega}(q) < +\infty$, since $|\Omega| = \pi \sum\limits_{n=1}^{+\infty} n|a_n|^2$. Moreover
$\gamma(q) = \frac{1}{2\pi}\log(|a_1|)$ and $\left.\frac{\partial R(w,q)}{\partial w}\right|_{w=q} = \frac{a_2}{4\pi a_1^2}$.
Then, see [S.Y.A. Chang, C.C. Chen & C.S. Lin **(2003)**] and [D.B. & C.S. Lin **(2014)**], Ω is of second kind if and only if there exists a maximum point q of γ such that $D_{\Omega}(q) > 0$.

Also, if $D_{\Omega}(p) \leq 0$ for a critical point p of γ, then p is a maximum point and in particular it is the unique maximum point and it is non degenerate (in the sense that $\det(D^2\gamma)(p) \neq 0$).

Therefore Ω is of first kind if and only if there exists a a critical point p of γ such that $D_{\Omega}(p) \leq 0$. In particular, if γ has more than one maximum point, then Ω is of second kind.

Appendix: domains of first/second kind.

The reason why $D_\Omega(q)$ plays such a role is that, see [S.Y.A. Chang, C.C. Chen & C.S. Lin **(2003)**], if v_n is a blow up sequence for $(\mathbf{Q}_{\mu,\Omega})$ as $\lambda_n = \mu_n \int\limits_\Omega e^{v_n} \to 8\pi$, or equivalently if ψ_n is a blow up sequence for $(\mathbf{P}_{\lambda,\Omega})$ as $\lambda_n \to 8\pi$, then

$$\lambda_n - 8\pi = \varepsilon_n (D_\Omega(q) + \mathrm{o}(1)), \quad \varepsilon_n \to 0, \text{ as } n \to +\infty.$$

The fact that the set of existence (w.r.t. μ) of solutions is an interval is always true if the nonlinearity is positive, increasing and continuous, and in particular this is sufficient to ensure the monotonicity of v_{μ} along the branch of minimal solutions [Keller & Cohen **(1967)**]. Besides, it was shown in [Keller & Cohen **(1967)**], [Crandall & Rabinowitz **(1975)**] that the first eigenvalue of the linearized equation for $\mathbf{(Q_{\mu,\Omega})}$ at (μ, v_{μ}) (say $\nu_1(\mu)$) is strictly positive if and only if v_{μ} is minimal. Therefore this is an alternative characterization of minimal solutions.
Also, $\nu_1(\mu)$ is monotone decreasing in $(0, \mu_\star(\Omega))$ along the branch of minimal solutions and obviously $\nu_1(\mu_\star(\Omega)) = 0$.
Another classical result [Fujita **(1974)**], [Keener & Keller **(1974)**], is the non existence of fully ordered triples in $(0, \mu_\star(\Omega))$.

Canonical and Microcanonical Variational Principles.

By defining the entropy and the energy of a density ρ,

$$\mathfrak{S}(\rho) = -\int\limits_{\Omega} \rho \log(\rho), \quad \mathcal{E}(\rho) = \frac{1}{2}\int\limits_{\Omega} \rho G[\rho],$$

then, setting $\frac{1}{\kappa T} = \beta = -\lambda$, solutions of the mean field equation $(\mathbf{P}_{\lambda,\Omega})$ arise (Caglioti, Lions, Marchioro & Pulvirenti **(1995)**) as critical densities of the canonical variational principle,

$$\begin{cases} \inf\left\{-\frac{1}{\beta}\mathfrak{S}(\rho) + \mathcal{E}(\rho),\ \rho \geq 0 \text{ a. e.}, \int\limits_{\Omega}\rho = 1\right\}, & \beta > 0 \\ \sup\left\{-\frac{1}{\beta}\mathfrak{S}(\rho) + \mathcal{E}(\rho),\ \rho \geq 0 \text{ a. e.}, \int\limits_{\Omega}\rho = 1\right\}, & -8\pi < \beta < 0 \end{cases}$$

or either of the microcanonical variational principle,

$$\sup\left\{\mathfrak{S}(\rho) \,|\, \mathcal{E}(\rho) = E,\ \rho \geq 0 \text{ a. e.}, \int\limits_{\Omega}\rho = 1\right\}, \quad E \in (0, +\infty).$$