Lorentz Force Equation and Mountain Pass Theorem

David ARCOYA

Universidad de Granada

Incontro su Variational methods, with applications to problems in mathematical physics and geometry
30 November - 1 December 2019, Istituto Canossiano San Trovaso, Venezia
Per festeggare i 75 anni di Antonio Ambrosetti!

Supported by FEDER-MINECO PGC2018-096422-B-100 and Junta de Andalucía FQM-116

JOURNAL OF FUNCTIONAL ANALYSIS 14, 349–381 (1973)

Dual Variational Methods in Critical Point Theory and Applications

ANTONIO AMBROSETTI*

Scuola Normale Superiore, Pisa, Italy

AND

PAUL H. RABINOWITZ†

Istituto di Matematica "L. Tonelli," Università di Pisa, Pisa—Italy and Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

Communicated by J. L. Lions

Received April 5, 1973

This paper contains some general existence theorems for critical points of a continuously differentiable functional I on a real Banach space. The strongest results are for the case in which I is even. Applications are given to partial differential and integral equations.

This paper develops dual variational methods to prove the existence and estimate the number of critical points possessed by a real valued continuously differentiable functional I on a real Banach space E. Our strongest results are obtained for the case in which I is even. I need neither be bounded from above nor below. This study was motivated by existence questions for nonlinear elliptic partial differential equations and several applications in this direction will be given as well as to integral equations.

To illustrate the sort of situation treated here, suppose I is even with $I(0) = 0$ and $I > 0$ near 0 in some uniform fashion. Then $u = 0$ is a local minimum for I. If I is also negative near ∞ when restricted to finite dimensional subspaces of E and satisfies a version of the

Joint works with

- Cristian Bereanu (Institute of Mathematics “Simion Stoilow”, Romanian Academy, Romania.)
- Pedro J. Torres (Universidad de Granada, Spain).

Lorentz force equation

The relativistic motion $q(t) = (q_1(t), q_2(t), q_3(t))$ of a charged particle in a electromagnetic field

- with E electric field and B magnetic field, and
- the mass-to-charge ratio and the speed of light are equal to one

satisfies the Lorentz force equation

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = E(t,q) + q' \times B(t,q), \tag{LFE}$$

Lorentz force equation

The relativistic motion $q(t) = (q_1(t), q_2(t), q_3(t))$ of a charged particle in a electromagnetic field

- with E electric field and B magnetic field, and
- the mass-to-charge ratio and the speed of light are equal to one

satisfies the Lorentz force equation

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = E(t,q) + q' \times B(t,q), \tag{LFE}$$

By a solution q of the LFE we mean a function $q = (q_1, q_2, q_3)$ of class C^2 such that $|q'(t)| < 1$ for all t, and which verifies the equation.

Lorentz force equation

Historically, the LFE dates back to

- Lorentz 1904
- H. Poincaré, 1906.
- M. Planck, 1906.

and can be found in many textbooks and monographies on Classical Mechanics and Electrodynamics, see for instance

- J.D. Jackson, Classical Electrodynamics, Third edition, Wiley, 1999. (Chapter 12)
- L.D. Landau, E.M. Lifschitz, The Classical Theory of Fields, Fourth Edition: Volume 2, Butterworth-Heinemann, 1980. (Chapter 3).

Most of the studies on the dynamics of LFE are limited to the identification of exact solutions for particular cases of simple electromagnetic fields:

- uniform and static fields in the book Landau & Lifschitz,
- circular, linear or elliptically polarized electromagnetic waves in Acharya-Saxena '93, Andreev-Makarov-Rukhadze '09, Shebalin '88.
- ...

A principle of least action for the relativistic case?

In his 1906 paper (Sections 2 and 7), Poincaré identifies formally the LFE as the Euler-Lagrange equation associated to the relativistic Lagrangian

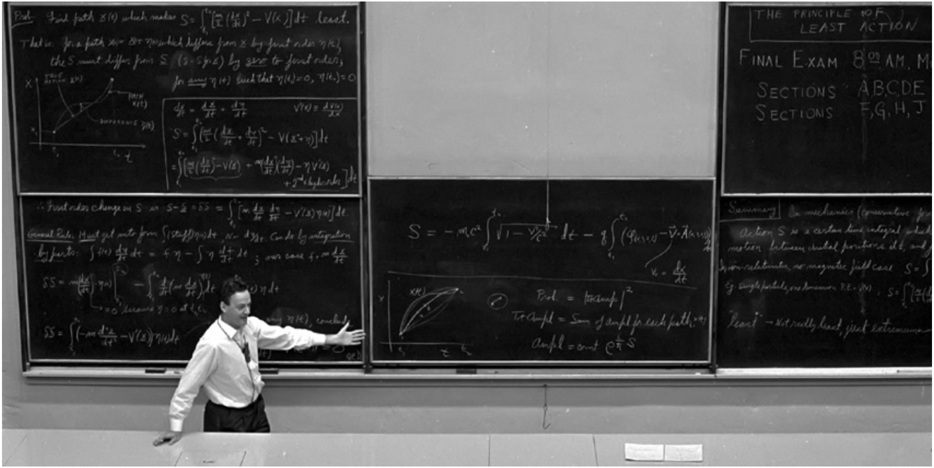
$$\mathcal{L}(t, q, q') = 1 - \sqrt{1 - |q'|^2} + q' \cdot W(t, q) - V(t, q),$$

where $V : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ and $W : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ are the electric and magnetic potentials respectively; i.e.,

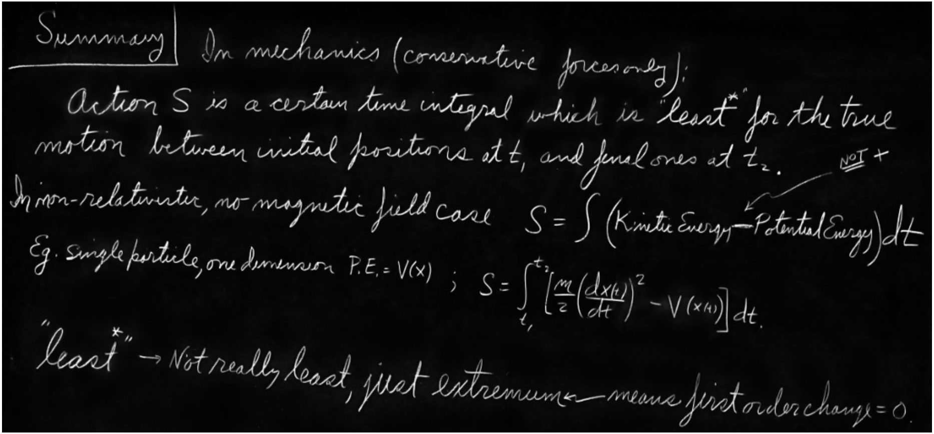
$$E = -\nabla_q V - \frac{\partial W}{\partial t}, \qquad B = \mathrm{curl}_q\, W,$$

Independently, the relativistic Lagrangian is also given by Planck.

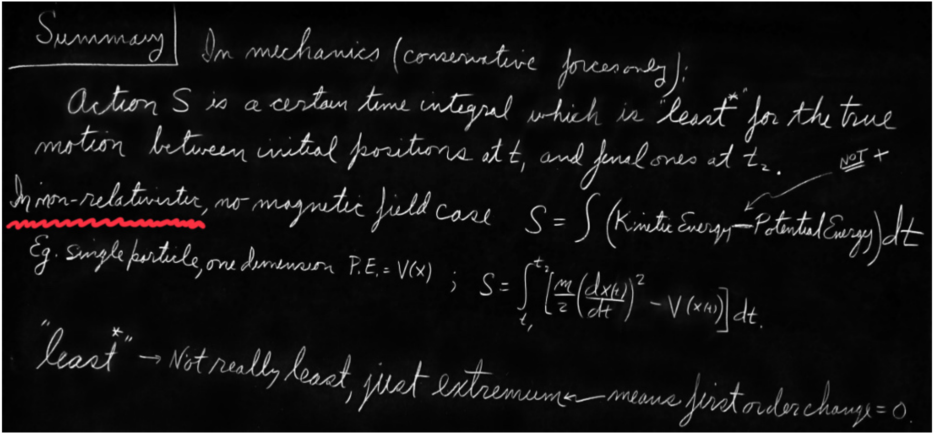
A motivation - 19^{th} lecture of Feynman '63



A motivation - 19^{th} lecture of Feynman '63



A motivation - 19^{th} lecture of Feynman '63



19^{th} lecture of Feynman '63

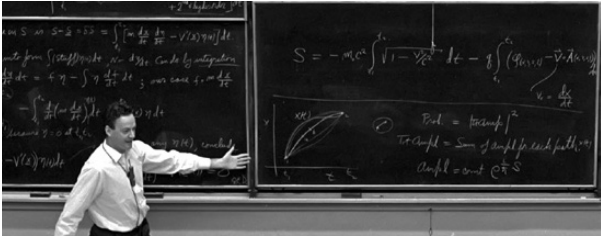
"Is there a corresponding principle of least action for the relativistic case? There is. The formula in the case of relativity is the following:"

$$\mathcal{I}(q) := \int_0^T \mathcal{L}(t, q, q')dt$$

where the relativistic Lagrangian is

$$\mathcal{L}(t, q, p) = 1 - \sqrt{1 - |p|^2} + p \cdot W(t, q) - V(t, q).$$

"I will leave to the more ingenious of you the problem to demonstrate that this action formula does, in fact, give the correct equations of motion for relativity. May I suggest you do it first without the W, that is, for no magnetic field?"



19^{th} lecture of Feynman '63

"*Is there a corresponding principle of least action for the relativistic case? There is. The formula in the case of relativity is the following:*"

$$\mathcal{I}(q) := \int_0^T \mathcal{L}(t, q, q')dt$$

where the relativistic Lagrangian is

$$\mathcal{L}(t, q, p) = 1 - \sqrt{1 - |p|^2} + p \cdot W(t, q) - V(t, q).$$

"*I will leave to the more ingenious of you the problem to demonstrate that this action formula does, in fact, give the correct equations of motion for relativity. May I suggest you do it first without the W, that is, for no magnetic field?*"

In other words, according to Poincaré, Planck, Feynman the critical points of the action functional $\mathcal{I}$ should be the solutions of LFE

$$\left(\frac{q'}{\sqrt{1 - |q'|^2}}\right)' = E(t, q) + q' \times B(t, q),$$

C^1-critical point theory is not applicable!

However, the derivative of the action functional

$$\mathcal{I}(q) := \int_0^T \mathcal{L}(t, q, q')dt, \qquad \mathcal{L}(t, q, p) = 1 - \sqrt{1 - |p|^2} + p \cdot W(t, q) - V(t, q),$$

is not easy at all.

C^1-critical point theory is not applicable!

However, the derivative of the action functional

$$\mathcal{I}(q) := \int_0^T \mathcal{L}(t,q,q')dt, \qquad \mathcal{L}(t,q,p) = 1-\sqrt{1-|p|^2}+p\cdot W(t,q)-V(t,q),$$

is not easy at all. Observe that

- the second part

  $$L(t,q,p) = p \cdot W(t,q) - V(t,q), \quad (t,q,p) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$$

  is a smooth function that accounts for the effect of the fields on the particle,

C^1-critical point theory is not applicable!

However, the derivative of the action functional

$$\mathcal{I}(q) := \int_0^T \mathcal{L}(t,q,q')dt, \qquad \mathcal{L}(t,q,p) = 1-\sqrt{1-|p|^2}+p\cdot W(t,q)-V(t,q),$$

is not easy at all. Observe that

- the second part

  $$L(t,q,p) = p \cdot W(t,q) - V(t,q), \quad (t,q,p) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$$

  is a smooth function that accounts for the effect of the fields on the particle,
- while the term $1-\sqrt{1-|p|^2}$ is only defined for $|p| \leq 1$ and it is not differentiable!

This means that the action functional $\mathcal{I}$ is not of class C^1 and the usual C^1-critical point theory is not applicable.

C^1-critical point theory is not applicable!

This lack of regularity of the action functional is a typical situation inherent to problems involving the relativistic acceleration. It was solved for the first time by Brezis & Mawhin '10 for the case of a forced relativistic pendulum with periodic boundary conditions.

$$\left.\begin{array}{c} \left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' + A\sin q = h(t),\ t\in(0,T) \\[2ex] q(0)=q(T),\ q'(0)=q'(T) \end{array}\right\}$$

- The global minimizer of the action functional is in fact a solution by means of a suitable variational inequality.
- The problem considered is scalar and, in addition, $W\equiv 0$ in it.

C^1-critical point theory is not applicable!

Later, in Bereanu - Jebelean - Mawhin '11 (see also Bonheure -D'Avenia - Pomponio - Reichel '16) the action functional $\mathcal{I}$ is identified for the first time as the sum

$$\mathcal{I} = \Psi + \mathcal{F} \text{ with } W \equiv 0$$

with

- Ψ a proper convex lower semicontinuous functional in the space of the continuous functions $C([0,T],\mathbb{R}^3)$,
- and $\mathcal{F}$ a C^1-functional in $C([0,T],\mathbb{R}^3)$.

$\Longrightarrow$ $\boxed{\text{Szulkin '86 critical point theory is applicable.}}$

Indeed,

$$W = 0 \implies \mathcal{F}(q) = \int_0^T L(t,q,q')dt = -\int_0^T V(t,q)dt$$

can be defined in $C([0,T],\mathbb{R}^3)$ and then it is not difficult to prove that every Palais-Smale sequence (in Szulkin sense) admits a subsequence converging in that space.

C^1-critical point theory is not applicable!

In our case, the presence of a magnetic potential ($W \not\equiv 0$) requires a complete reformulation from the very beginning.

C^1-critical point theory is not applicable!

In our case, the presence of a magnetic potential ($W \not\equiv 0$) requires a complete reformulation from the very beginning.

Indeed, in this case,

- the functional is not properly defined in the continuous functions space $C([0,T],\mathbb{R}^3)$,
- the notion of Palais-Smale is not clear at all.

The functional framework

Let $T > 0$ be fixed. If $W_0^{1,\infty}(0,T)$ denotes the space of all Lipschitz real functions in $[0,T]$ (or equivalently the absolutely continuous functions in $[0,T]$ with bounded derivatives) which vanishes at 0 and T, we consider the Banach space

$$W_0^{1,\infty} = [W_0^{1,\infty}(0,T)]^3$$

endowed with the usual norm $\|\cdot\|_{1,\infty}$ given by

$$\|q\|_{1,\infty} = \|q\|_\infty + \|q'\|_\infty \quad (q \in W_0^{1,\infty}),$$

where $\|q\|_\infty = \max_{t\in[0,T]} |q(t)|$ and $\|q'\|_\infty = \max_{t\in[0,T]} |q'(t)|$.

The funct. framework - "Smooth" part of the *relat. Lagr.* $\mathcal{L}$

If the potentials $V, W \in C^1$, then the action functional $\mathcal{F} : W_0^{1,\infty} \to \mathbb{R}$ (associated to the "smooth" part

$$L(t,q,p) = p \cdot W(t,q) - V(t,q) \qquad ((t,q,p) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3),$$

of the *relativistic Lagrangian* $\mathcal{L}$) given by

$$\mathcal{F}(q) := \int_0^T L(t,q,q')dt = \int_0^T [q' \cdot W(t,q) - V(t,q)]dt, \qquad \forall q \in W_0^{1,\infty}$$

is C^1 with

$$\mathcal{F}'(q)[\varphi] = \int_0^T (\mathcal{E}(t,q,q') - \nabla_q V(t,q)) \cdot \varphi dt + \int_0^T W(t,q) \cdot \varphi' dt, \ \forall q, \varphi \in W_0^{1,\infty},$$

where

$$\mathcal{E}(t,q,p) = (p \cdot D_{q_1} W(t,q), p \cdot D_{q_2} W(t,q), p \cdot D_{q_3} W(t,q).$$

The functional framework - “nonsmooth” part

$$\mathcal{K} = \{q \in W_0^{1,\infty} : \|q'\|_\infty \leq 1\},$$

$$\Phi(s) = 1 - \sqrt{1 - s^2} \qquad (s \in [-1, 1]),$$

The action functional corresponding to the “nonsmooth” part of the relativistic Lagrangian:

$$\Psi : W_0^{1,\infty} \to (-\infty, +\infty],$$

$$\Psi(q) = \begin{cases} \displaystyle\int_0^T \Phi(q')dt = \int_0^T [1 - \sqrt{1 - |q'|^2}]dt, & \text{if } q \in \mathcal{K}, \\ +\infty, & \text{if } \ q \notin W_0^{1,\infty} \setminus \mathcal{K}. \end{cases}$$

The functional framework - “nonsmooth” part

$$\mathcal{K} = \{q \in W_0^{1,\infty} : \|q'\|_\infty \leq 1\},$$

$$\Phi(s) = 1 - \sqrt{1 - s^2} \qquad (s \in [-1, 1]),$$

The action functional corresponding to the “nonsmooth” part of the relativistic Lagrangian:

$$\Psi : W_0^{1,\infty} \to (-\infty, +\infty],$$

$$\Psi(q) = \begin{cases} \displaystyle\int_0^T \Phi(q')dt = \int_0^T [1 - \sqrt{1 - |q'|^2}]dt, & \text{if } q \in \mathcal{K}, \\ +\infty, & \text{if } \ q \notin W_0^{1,\infty} \setminus \mathcal{K}. \end{cases}$$

Lemma

The restriction of the functional Ψ to its domain $\mathcal{K}$ is continuous. □

The functional framework - "nonsmooth" part

$$\mathcal{K} = \{q \in W_0^{1,\infty} : \|q'\|_\infty \leq 1\},$$

Lemma (Brezis & Mawhin '10)

(i) *$\mathcal{K}$ is convex and closed in $C([0,T],\mathbb{R}^3)$ and thus in $W_0^{1,\infty}$. Moreover,*

$$\mathcal{K} \supset (q_n(t)) \longrightarrow q(t) \in C([0,T]),\ \forall t \in [0,T] \implies \begin{cases} q \in \mathcal{K} \\ q_n' \xrightarrow[\sigma(L^\infty, L^1)]{w^*} q'. \end{cases}$$

(ii) *If (q_n) is a sequence in $\mathcal{K}$ converging in $C([0,T],\mathbb{R}^3)$ to q, then*

$$\Psi(q) \leq \liminf_{n\to\infty} \Psi(q_n).$$

In particular, the functional Ψ is weakly lower semicontinuous and convex in $W_0^{1,\infty}$. □

The functional framework - action functional

Consider the Euler-Lagrange action functional associated to the relativistic Lagrangian $\mathcal{L}$ with zero Dirichlet boundary conditions, i.e.,

$$\mathcal{I} : W_0^{1,\infty} \to (-\infty, +\infty], \quad \mathcal{I} = \Psi + \mathcal{F}.$$

- Ψ is proper convex l. s. c. functional,
- $\mathcal{F} \in C^1(W_0^{1,\infty})$.

Definition (Szulkin '86)

A function $q \in W_0^{1,\infty}$ is a **critical point** of $\mathcal{I}$ if $q \in \mathcal{K}$ and

$$\Psi(\varphi) - \Psi(q) + \mathcal{F}'(q)[\varphi - q] \geq 0 \quad \text{for all } \varphi \in W_0^{1,\infty}, \text{ i.e.}$$

$$\int_0^T [\sqrt{1 - |q'|^2} - \sqrt{1 - |\varphi'|^2}]dt + \int_0^T [\mathcal{E}(t, q, q') - \nabla_q V(t, q)] \cdot (\varphi - q)dt$$
$$+ \int_0^T W(t, q) \cdot (\varphi' - q')dt \geq 0, \quad \text{for all } \varphi \in \mathcal{K}.$$

The functional framework

Theorem

If $V : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ *and* $W : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ *are two* C^1*-functions, then a function* $q \in W_0^{1,\infty}$ *is a critical point of* $\mathcal{I}$ *if and only if* q *is a solution of the Lorentz force equation with zero Dirichlet boundary conditions on* $[0,T]$, *i.e.*

$$\left(\frac{q'}{\sqrt{1-|q'|^2}} \right)' + (W(t,q))' = \mathcal{E}(t,q,q') - \nabla_q V(t,q),$$

$$q(0) = 0 = q(T).$$

Principle of least action for Dirichlet problems at the end

Theorem (Principle of the least action for LFE with 0-Dirichlet b.c.)

The Lagrangian action $\mathcal{I}$ associated to Lorentz equation with zero Dirichlet boundary conditions is bounded from below and attains its infimum at some $q \in \mathcal{K}$, which is a solution of LFE with zero boundary conditions.

Principle of least action for Dirichlet problems at the end

Theorem (Principle of the least action for LFE with 0-Dirichlet b.c.)

The Lagrangian action $\mathcal{I}$ associated to Lorentz equation with zero Dirichlet boundary conditions is bounded from below and attains its infimum at some $q \in \mathcal{K}$, which is a solution of LFE with zero boundary conditions.

Proof. Let (q_n) be a minimizing sequence of $\mathcal{I}$, that is

$$(q_n) \subset \mathcal{K}, \quad \mathcal{I}(q_n) \to \inf_{W_0^{1,\infty}} \mathcal{I} = \inf_{\mathcal{K}} \mathcal{I} \text{ as } n \to \infty.$$

Principle of least action for Dirichlet problems at the end

Theorem (Principle of the least action for LFE with 0-Dirichlet b.c.)

The Lagrangian action $\mathcal{I}$ associated to Lorentz equation with zero Dirichlet boundary conditions is bounded from below and attains its infimum at some $q \in \mathcal{K}$, which is a solution of LFE with zero boundary conditions.

Proof. Let (q_n) be a minimizing sequence of $\mathcal{I}$, that is

$$(q_n) \subset \mathcal{K}, \quad \mathcal{I}(q_n) \to \inf_{W_0^{1,\infty}} \mathcal{I} = \inf_{\mathcal{K}} \mathcal{I} \text{ as } n \to \infty.$$

$$\left.\begin{array}{l} q_n(t) = \int_0^t q_n'(s)ds \\ \\ |q_n'(s)| \leq 1 \end{array}\right\} \Longrightarrow \|q_n\|_\infty \leq T \Longrightarrow \boxed{\|q_n\|_{1,\infty} \leq T+1} \quad \forall n \in \mathbb{N}.$$

Principle of least action for Dirichlet problems at the end

Theorem (Principle of the least action for LFE with 0-Dirichlet b.c.)

The Lagrangian action $\mathcal{I}$ associated to Lorentz equation with zero Dirichlet boundary conditions is bounded from below and attains its infimum at some $q \in \mathcal{K}$, which is a solution of LFE with zero boundary conditions.

Proof. Let (q_n) be a minimizing sequence of $\mathcal{I}$, that is

$$(q_n) \subset \mathcal{K}, \quad \mathcal{I}(q_n) \to \inf_{W_0^{1,\infty}} \mathcal{I} = \inf_{\mathcal{K}} \mathcal{I} \text{ as } n \to \infty.$$

$$\left.\begin{array}{l} q_n(t) = \int_0^t q_n'(s)ds \\ \\ |q_n'(s)| \le 1 \end{array}\right\} \Longrightarrow \|q_n\|_\infty \le T \Longrightarrow \boxed{\|q_n\|_{1,\infty} \le T+1} \quad \forall n \in \mathbb{N}.$$

By using $W_0^{1,\infty} \overset{\text{compact}}{\hookrightarrow} C([0,T],\mathbb{R}^3)$, up to a subsequence

$$\exists q \in C([0,T],\mathbb{R}^3) \text{ such that } \|q_n - q\|_\infty \to 0.$$

Principle of least action for Dirichlet problems at the end

By Lemma of Brezis-Mawhin we deduce that

$$q \in \mathcal{K}, \quad \Psi(q) \leq \liminf_{n\to\infty} \Psi(q_n), \quad \lim_{n\to\infty} \int_0^T V(t, q_n)dt = \int_0^T V(t,q)dt,$$

$$\lim_{n\to\infty} \int_0^T (q_n' - q') \cdot \varphi dt = 0 \quad \forall \varphi \in L^1 \text{ (w^*-convergence).}$$

Consequently

$$\begin{aligned}\lim_{n\to\infty} \int_0^T q_n' \cdot W(t, q_n)dt &= \lim_{n\to\infty} \int_0^T q_n' \cdot (W(t,q_n) - W(t,q))dt \\ &\quad + \lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q)dt. \\ &= \int_0^T q' \cdot W(t,q)dt\end{aligned}$$

$\implies \mathcal{I}(q) = \inf_{\mathcal{K}} \mathcal{I}$, i.e., $\mathcal{I}$ attains its infimum at $q \in \mathcal{K}$. Since every local minimizer of $\mathcal{I}$ is a Szulkin-critical point, q is also a solution of LFE with zero Dirichlet boundary conditions. $\square$

Principle of least action for Dirichlet problems

As a simple application of the above principle we have the following result of existence of a nonzero solution.

Principle of least action for Dirichlet problems

As a simple application of the above principle we have the following result of existence of a nonzero solution.

Theorem

If there exist $1 \leq \mu < \min\{\nu, 2\}$ *and* $d > 0$ *such that*

$$V(t,q) \geq d|q|^{\mu} + V(t,0) \text{ and } |W(t,q)| \leq d|q|^{\nu}, \ \forall t \in [0,T] \ \forall |q| \leq T,$$

then LFE has at least one nonzero solution in $W_0^{1,\infty}$ *which is a minimizer of the Lagrangian action* $\mathcal{I}$.

Principle of least action for Dirichlet problems

As a simple application of the above principle we have the following result of existence of a nonzero solution.

Theorem

If there exist $1 \leq \mu < \min\{\nu, 2\}$ *and* $d > 0$ *such that*

$$V(t,q) \geq d|q|^{\mu} + V(t,0) \text{ and } |W(t,q)| \leq d|q|^{\nu}, \ \forall t \in [0,T] \ \forall |q| \leq T,$$

then LFE has at least one nonzero solution in $W_0^{1,\infty}$ *which is a minimizer of the Lagrangian action* $\mathcal{I}$.

Proof. Since $W(t,0) = 0$ for every $t \in [0,T]$, we observe that

$$\mathcal{I}(0) = -\int_0^T V(t,0)dt.$$

If $q_0 \in \mathcal{K} \setminus \{0\}$, then $\epsilon q_0 \in \mathcal{K}$ for all $\epsilon \in (0,1]$. In addition,

$$\int_0^T \left[1 - \sqrt{1 - |\epsilon q_0'|^2}\right] dt \leq \int_0^T |\epsilon q_0'|^2 dt.$$

Principle of least action for Dirichlet problems

Since $\|q\|_\infty \leq T$, $\forall q \in \mathcal{K}$, we deduce from the hypothesis the existence of positive constants $C_1, C_2 > 0$ such that

$$\mathcal{I}(\epsilon q_0) \leq C_1(\epsilon^2 + \epsilon^\nu) - C_2\epsilon^\mu - \int_0^T V(t,0)dt.$$

In particular, for ϵ small enough we have $\mathcal{I}(\epsilon q_0) < \mathcal{I}(0)$, which implies that

$$\inf_{W_0^{1,\infty}} \mathcal{I} < \mathcal{I}(0).$$

Using the principle of least action for Dirichlet problem, $\mathcal{I}$ attains its infimum at some $q \in \mathcal{K} \setminus \{0\}$, which is a nonzero solution of LFE. $\square$

Solutions via mountain pass theorem

Theorem

Assume that $E_\lambda = -\lambda \nabla_q V$ *and* $B = \mathit{curl}_q W$ *and* $\exists \mu, \nu > 2$, $\exists d > 0$ *s.t.*

$$V(t,q) \le d|q|^\mu, \quad |W(t,q)| \le d|q|^\nu, \ \forall t \in [0,T], \ \forall |q| \le T.$$

$\exists q_0 \in \mathcal{K} \setminus \{0\} \, : \, \int_0^T V(t,q_0)dt > 0 \implies \exists \lambda^* > 0$ *such that* $\forall \lambda \ge \lambda^*$, *problem*

$$\left(\frac{q'}{\sqrt{1-|q'|^2}} \right)' = E_\lambda(t,q) + q' \times B(t,q), \quad q(0) = 0 = q(T),$$

possesses at least one nonzero solution.
If in addition $q = 0$ *is solution, i.e., if* $V(t,0) = 0$ *for every* $t \in [0,T]$, *then there exists also a second nonzero solution.*

Proof - First solution

$$\mathcal{I}_{\lambda}(q) = \Psi(q) + \int_0^T q' \cdot W(t,q)dt - \lambda \int_0^T V(t,q)dt$$

We have

$$\mathcal{I}_{\lambda}(q_0) = \Psi(q_0) + \int_0^T q_0' \cdot W(t,q_0)dt - \lambda \underbrace{\int_0^T V(t,q_0)dt}_{>0}$$

$$\implies \qquad \exists \lambda^* >> 0 \text{ such that } \inf_{q \in \mathcal{K}} \mathcal{I}_{\lambda}(q) \le \mathcal{I}_{\lambda}(q_0) < 0, \quad \forall \lambda \ge \lambda^*.$$

On the other hand, the hypothesis

$$V(t,q) \le d|q|^{\mu} \implies V(t,0) \le 0, \ \forall t \in [0,T] \implies \mathcal{I}_{\lambda}(0) \ge 0$$

Using the principle of least action, $\mathcal{I}$ attains its infimum at some $q_* \in \mathcal{K} \setminus \{0\}$, which is a nonzero solution.

Proof - Second solution for $\lambda \geq \lambda^*$

- $V(t,0) = 0 \ \forall t \in [0,T] \implies \mathcal{I}_\lambda(0) = 0 > \mathcal{I}_\lambda(q_*)$.
- $1 - \sqrt{1-s^2} \geq \frac{s^2}{2}, \ \forall s \in [-1,1] \overset{q \in \mathcal{K}}{\implies} \Psi(q) \geq \frac{1}{2}\int_0^T |q'|^2 dt = \frac{1}{2}\|q\|_{H_0^1}^2$.
- $V(t,q) \leq d|q|^\mu \implies \int_0^T V(t,q)dt \leq C\|q\|_{L^\mu}^\mu$.
- $\left.\begin{array}{c} \|q'\|_\infty \leq 1 \\ \\ |W(t,q)| \leq d|q|^\nu \end{array}\right\} \implies \left|\int_0^T q' \cdot W(t,q)dt\right| \leq C\|q\|_{L^\nu}^\nu$

Therefore,

$$\mathcal{I}_\lambda(q) \geq \frac{1}{2}\|q\|_{H_0^1}^2 - C\|q\|_{L^\nu}^\nu - \lambda C\|q\|_{L^\mu}^\mu, \quad \forall q \in \mathcal{K}.$$

Proof - Second solution for $\lambda \geq \lambda^*$

$$\mathcal{I}_\lambda(q) \geq \frac{1}{2}\|q\|_{H_0^1}^2 - C\|q\|_{L^\nu}^\nu - \lambda C\|q\|_{L^\mu}^\mu, \quad \forall q \in \mathcal{K}.$$

Using that $H_0^1 \hookrightarrow L^\mu \cap L^\nu$ (since $\mu, \nu > 2$),

$$\mathcal{I}_\lambda(q) \geq \frac{1}{2}\|q\|_{H_0^1}^2 - C\|q\|_{H_0^1}^\nu - \lambda C\|q\|_{H_0^1}^\mu, \quad \forall q \in \mathcal{K},$$

$\implies \exists r \in (0, \|q_*\|_{H_0^1})$ & $\exists \alpha > 0$ s. t. $\mathcal{I}_\lambda(q) \geq \alpha$, $\forall q \in \mathcal{K}$ with $\|q\|_{H_0^1} = r$.

Let $\Gamma = \{\gamma : [0,1] \to W_0^{1,\infty} \ : \gamma$ cont., $\gamma(0) = 0,\ \gamma(1) = q_*\}$.

$$W_0^{1,\infty} \hookrightarrow H_0^1 \implies \forall \gamma \in \Gamma,\ \exists t_0 \in [0,1] \text{ s.t. } \|\gamma(t_0)\|_{H_0^1} = r.$$

Consequently,

$$\sup_{t \in [0,1]} \mathcal{I}_\lambda(\gamma(t)) \geq \alpha > \mathcal{I}_\lambda(0) = 0 > \mathcal{I}_\lambda(q_*), \quad \forall \gamma \in \Gamma,\ \forall \lambda \geq \lambda^*$$

i.e. $\mathcal{I}_\lambda$ satisfies the geometry of Mountain Pass.

Theorem (Non-smooth mountain pass theorem)

Assume E is a Banach space and $\mathcal{I} = \Psi + \mathcal{F} : E \to (-\infty, +\infty]$ where

(i) *Ψ is a convex and proper functional with a closed domain $\mathrm{Dom}\,\Psi := \{v \in E \,:\, \Psi(v) < \infty\}$ in E and Ψ is continuous in $\mathrm{Dom}\,\Psi$.*

(ii) *$\mathcal{F} : E \to \mathbb{R}$ is a C^1-functional.*

Let $q_* \in E \setminus \{0\}$, $\Gamma = \{\gamma : [0,1] \to E : \gamma$ is cont. $\gamma(0) = 0$, $\gamma(1) = q_*\}$. If

$$c_1 := \max\{\mathcal{I}(0), \mathcal{I}(q_*)\} < c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{I}(\gamma(t)) < \infty,$$

$\implies$ $\forall \varepsilon > 0,\ \forall \gamma \in \Gamma$ *such that* $c \le \max_{t\in K} \mathcal{I}(\gamma(t)) \le c + \frac{\varepsilon}{2}$, *there exist* $\gamma_\varepsilon \in \Gamma$ *and* $q_\varepsilon \in \gamma_\varepsilon(K) \subset E$ *satisfying*

$$c \le \max_{t\in K} \mathcal{I}(\gamma_\varepsilon(t)) \le \max_{t\in K} \mathcal{I}(\gamma(t)) \le c + \frac{\varepsilon}{2},$$

$$\max_{t\in K} \|\gamma_\varepsilon(t) - \gamma(t)\| \le \sqrt{\varepsilon}, \qquad c - \varepsilon \le \mathcal{I}(q_\varepsilon) \le c + \frac{\varepsilon}{2},$$

$$\Psi(\varphi) - \Psi(q_\varepsilon) + \mathcal{F}'(q_\varepsilon)[\varphi - q_\varepsilon] \ge -\sqrt{\varepsilon}\|\varphi - q_\varepsilon\| \quad \forall \varphi \in E.$$

Remark

- Notice that the continuity of Ψ in its closed domain implies that Ψ is lower semicontinuous in E. This allows us to follows the ideas of A & Boccardo '96.
- A similar theorem of mountain-pass type is proved in the very recent paper Alves & de Morais Filho '18 without the continuity condition.

Proof - Cont.

Applying this non-smooth version of the mountain pass theorem with $\varepsilon_n = \frac{1}{n}$, we obtain the existence of a sequence $(q_n) \subset W_0^{1,\infty}$ such that

$$\lim_{n\to\infty} \mathcal{I}_\lambda(q_n) = c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) \geq \alpha$$

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\frac{1}{n}\|\varphi - q_n\|_{1,\infty}, \quad \forall \varphi \in \mathcal{K}.$$

Proof - Cont.

Applying this non-smooth version of the mountain pass theorem with $\varepsilon_n = \frac{1}{n}$, we obtain the existence of a sequence $(q_n) \subset W_0^{1,\infty}$ such that

$$\lim_{n\to\infty} \mathcal{I}_\lambda(q_n) = c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) \geq \alpha$$

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\frac{1}{n}\|\varphi - q_n\|_{1,\infty}, \quad \forall\varphi \in \mathcal{K}.$$

By using

Lemma (Compactness condition)

If $(q_n) \subset W_0^{1,\infty}$ *&* $(\varepsilon_n) \to 0$ *satisfying that*

$$\lim_{n\to\infty} \mathcal{I}_\lambda(q_n) = c \in \mathbb{R}$$

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\varepsilon_n\|\varphi - q_n\|_{1,\infty}, \quad \forall\varphi \in \mathcal{K},$$

then there exists a subsequence (q_{n_k}) *of* (q_n) *converging in* $C([0,T],\mathbb{R}^3)$ *to a critical point* $q \in \mathcal{K}$ *of* $\mathcal{I}_\lambda$ *with level* $\mathcal{I}_\lambda(q) = c$.

Proof - Cont.

Applying this non-smooth version of the mountain pass theorem with $\varepsilon_n = \frac{1}{n}$, we obtain the existence of a sequence $(q_n) \subset W_0^{1,\infty}$ such that

$$\lim_{n\to\infty} \mathcal{I}_\lambda(q_n) = c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) \geq \alpha$$

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\frac{1}{n}\|\varphi - q_n\|_{1,\infty}, \quad \forall \varphi \in \mathcal{K}.$$

$\implies \exists q^* \in \mathcal{K}$ critical point of $\mathcal{I}_\lambda$ with level $\mathcal{I}_\lambda(q^*) = c$.

Proof - Cont.

Applying this non-smooth version of the mountain pass theorem with $\varepsilon_n = \frac{1}{n}$, we obtain the existence of a sequence $(q_n) \subset W_0^{1,\infty}$ such that

$$\lim_{n\to\infty} \mathcal{I}_\lambda(q_n) = c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) \geq \alpha$$

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\frac{1}{n}\|\varphi - q_n\|_{1,\infty}, \quad \forall \varphi \in \mathcal{K}.$$

$\implies \exists q^* \in \mathcal{K}$ critical point of $\mathcal{I}_\lambda$ with level $\mathcal{I}_\lambda(q^*) = c$.

$$c \geq \alpha > 0 \implies q^* \neq 0,\ q^* \neq q_*.$$

Therefore, q^* is a second nonzero solution of LFE. $\square$

Symmetric functionals and T-periodic solutions

$$\mathcal{I}_* : W^{1,\infty}_* \longrightarrow (-\infty, +\infty], \quad \mathcal{I}_* = \Psi_* + \mathcal{F}$$

$$\mathcal{K}_* = \{q \in W^{1,\infty}_* : \|q'\|_\infty \leq 1\},$$

$$\Psi_*(q) = \begin{cases} \displaystyle\int_0^T [1 - \sqrt{1 - |q'|^2}] dt, & \text{if } q \in \mathcal{K}_*, \\ +\infty, & \text{if } \ q \notin W^{1,\infty}_* \setminus \mathcal{K}_*. \end{cases}$$

Symmetric functionals and T-periodic solutions

Theorem

Assume that W is odd and V is even in the second variable. If $\mathcal{I}_*$ is bounded from below in a subspace $\widetilde{X}_{l-1}$ of codimension $l-1$ with $l \geq 1$ and satisfies (wPS)-condition and for some $k \geq l$

$(\mathcal{I}_1)$ *there exist a subspace X_k of $W_*^{1,\infty}$ with $\dim X_k = k$ and $r > 0$ such that $\mathcal{I}_*(q) < \mathcal{I}_*(0)$ for all $q \in X_k$ with $\|q\|_\infty = r$,*

then $\mathcal{I}_*$ possesses at least $k - l + 1$ distinct pairs of nontrivial critical points with negative levels.

Symmetric functionals and T-periodic solutions

Theorem

Assume that W is odd and V is even in the second variable. If $\mathcal{I}_*$ is bounded from below in a subspace $\widetilde{X}_{l-1}$ of codimension $l-1$ with $l \geq 1$ and satisfies (wPS)-condition and for some $k \geq l$

$(\mathcal{I}_1)$ *there exist a subspace X_k of $W_*^{1,\infty}$ with $\dim X_k = k$ and $r > 0$ such that $\mathcal{I}_*(q) < \mathcal{I}_*(0)$ for all $q \in X_k$ with $\|q\|_\infty = r$,*

then $\mathcal{I}_*$ possesses at least $k - l + 1$ distinct pairs of nontrivial critical points with negative levels.

This theorem is deduced as a particular case of a general theorem improving Theorem 4.3 of Szulkin '86 where the author assumes that

- $l = 1$ and,
- instead of (wPS), that the stronger Szulkin' version of the Palais-Smale (PS) holds true.

Symmetric functionals and T-periodic solutions

Example

Assume that W is odd and V is even in their second variable with $V(t,0) = 0$ for every $t \in [0,T]$. Suppose also that there exist $r_1 \in (0,1)$, $c, d > 0$ and $\mu, \nu > 0$ with $\mu < \min\{2, \nu + 1\}$ such that

$$|W(t,q)| \leq c|q|^{\nu}, \ V(t,q) \geq d|q|^{\mu} \quad \text{for } t \in [0,T], |q| \leq r_1.$$

If, in addition, there exist $\overline{\mu} > 1$, $C > 0$ and a sufficiently large $R > 0$ such that either

$$|W(t,q)| + V(t,q) \leq -C|q|^{\overline{\mu}}, \quad \text{for } t \in [0,T], |q| \geq R,$$

or

$$|W(t,q)| - V(t,q) \leq -C|q|^{\overline{\mu}}, \quad \text{for } t \in [0,T], |q| \geq R,$$

then the LFE has infinitely many pairs of nontrivial T-periodic solutions.

MP Th. for non-smooth functionals having $\mathbb{Z}_2$-symmetry

Theorem

Assume that W is odd and V is even in the second variable with $V(t,0) = 0 \; \forall t \in [0,T]$. If $\mathcal{I}_*$ satisfies (wPS)-cond. and for integers $\overline{k} < k$

$(\mathcal{I}_1)$ *there exist a subspace X_k of $W_*^{1,\infty}$ with dim $X_k = k$ and $r > 0$ such that $\mathcal{I}_*(q) < \mathcal{I}_*(0)$ for all $q \in X_k$ with $\|q\|_\infty = r$,*

$(\mathcal{I}_2)$ *there exist a subspace $\widetilde{X}_{\overline{k}}$ of $W_*^{1,\infty}$ with codim $\widetilde{X}_{\overline{k}} = \overline{k} < k$ and constants $\rho \in (0,r)$ and $\alpha > 0$ such that $\mathcal{I}_*(q) \geq \alpha$ for all $q \in \widetilde{X}_{\overline{k}}$ with $\|q\|_\infty = \rho$,*

then $\mathcal{I}_*$ has at least $k - \overline{k}$ distinct pairs of nontrivial critical points with positive levels.

MP Th. for non-smooth functionals having $\mathbb{Z}_2$-symmetry

Theorem

Assume that W is odd and V is even in the second variable with $V(t,0) = 0\ \forall t \in [0,T]$. If $\mathcal{I}_*$ satisfies (wPS)-cond. and for integers $\overline{k} < k$

$(\mathcal{I}_1)$ *there exist a subspace X_k of $W_*^{1,\infty}$ with dim $X_k = k$ and $r > 0$ such that $\mathcal{I}_*(q) < \mathcal{I}_*(0)$ for all $q \in X_k$ with $\|q\|_\infty = r$,*

$(\mathcal{I}_2)$ *there exist a subspace $\widetilde{X}_{\overline{k}}$ of $W_*^{1,\infty}$ with codim $\widetilde{X}_{\overline{k}} = \overline{k} < k$ and constants $\rho \in (0,r)$ and $\alpha > 0$ such that $\mathcal{I}_*(q) \geq \alpha$ for all $q \in \widetilde{X}_{\overline{k}}$ with $\|q\|_\infty = \rho$,*

then $\mathcal{I}_*$ has at least $k - \overline{k}$ distinct pairs of nontrivial critical points with positive levels.

- Contrary to Theorem 4.4 of Szulkin, it is not required that $\mathcal{I}_*(q) \to -\infty$ when $\|q\|_{1,\infty} \to \infty$ with $q \in X_k$.

MP Th. for non-smooth functionals having $\mathbb{Z}_2$-symmetry

Theorem

Assume that W is odd and V is even in the second variable with $V(t,0) = 0\ \forall t \in [0,T]$. If $\mathcal{I}_*$ satisfies (wPS)-cond. and for integers $\overline{k} < k$

$(\mathcal{I}_1)$ *there exist a subspace X_k of $W_*^{1,\infty}$ with dim $X_k = k$ and $r > 0$ such that $\mathcal{I}_*(q) < \mathcal{I}_*(0)$ for all $q \in X_k$ with $\|q\|_\infty = r$,*

$(\mathcal{I}_2)$ *there exist a subspace $\widetilde{X}_{\overline{k}}$ of $W_*^{1,\infty}$ with codim $\widetilde{X}_{\overline{k}} = \overline{k} < k$ and constants $\rho \in (0,r)$ and $\alpha > 0$ such that $\mathcal{I}_*(q) \geq \alpha$ for all $q \in \widetilde{X}_{\overline{k}}$ with $\|q\|_\infty = \rho$,*

then $\mathcal{I}_*$ has at least $k - \overline{k}$ distinct pairs of nontrivial critical points with positive levels.

- Contrary to Theorem 4.4 of Szulkin, it is not required that $\mathcal{I}_*(q) \to -\infty$ when $\|q\|_{1,\infty} \to \infty$ with $q \in X_k$.
- We only impose the weaker (wPS) condition.

MP Th. for non-smooth functionals having $\mathbb{Z}_2$-symmetry

Theorem

Assume that W is odd and V is even in the second variable with $V(t,0) = 0$ $\forall t \in [0,T]$. If $\mathcal{I}_*$ satisfies (wPS)-cond. and for integers $\overline{k} < k$

- $(\mathcal{I}_1)$ *there exist a subspace X_k of $W_*^{1,\infty}$ with dim $X_k = k$ and $r > 0$ such that $\mathcal{I}_*(q) < \mathcal{I}_*(0)$ for all $q \in X_k$ with $\|q\|_\infty = r$,*
- $(\mathcal{I}_2)$ *there exist a subspace $\widetilde{X}_{\overline{k}}$ of $W_*^{1,\infty}$ with codim $\widetilde{X}_{\overline{k}} = \overline{k} < k$ and constants $\rho \in (0,r)$ and $\alpha > 0$ such that $\mathcal{I}_*(q) \geq \alpha$ for all $q \in \widetilde{X}_{\overline{k}}$ with $\|q\|_\infty = \rho$,*

then $\mathcal{I}_*$ has at least $k - \overline{k}$ distinct pairs of nontrivial critical points with positive levels.

- Contrary to Theorem 4.4 of Szulkin, it is not required that $\mathcal{I}_*(q) \to -\infty$ when $\|q\|_{1,\infty} \to \infty$ with $q \in X_k$.
- We only impose the weaker (wPS) condition.
- In contrast with the *"not fully satisfactory"* minimax characterization given in Szulkin, we give a satisfactory characterization.

MP Th. for non-smooth functionals having $\mathbb{Z}_2$-symmetry

Example

Let V be given by

$$V(t,q) = \lambda\beta(t)|q|^\mu \quad \text{for all } (t,q) \in [0,T] \times \mathbb{R}^3,$$

where $\mu > 2$ and $\beta : [0,T] \to \mathbb{R}$ is a positive, continuous function. If

$$W(t,\cdot) \text{ is odd } \forall t \in [0,T],$$

$$\lim_{|q|\to 0} \frac{|W(t,q)|}{|q|^2} = 0, \qquad \limsup_{|q|\to\infty} \frac{|W(t,q)|}{|q|^\mu} < \infty, \quad \text{unif. in } t \in [0,T],$$

then, for any integer $m \geq 1$, there is $\Lambda_m > 0$ such that the LFE has at least $2m$ pairs of nontrivial T-periodic solutions (m pairs corresponding to negative and m pairs to positive critical values of the relativistic Poincaré action functional) for any $\lambda \geq \Lambda_m$.

Grazie mille per l'attenzione.