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Density of chaotic dynamics in periodically forced pendulum-type equations

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Equazioni differenziali ordinarie. — *Density of chaotic dynamics in periodically forced pendulum-type equations.* Nota di Elena Bosetto, Enrico Serra e Susanna Terracini, presentata (*) dal Socio A. Ambrosetti.

Abstract. — We announce that a class of problems containing the classical periodically forced pendulum equation displays the main features of chaotic dynamics for a dense set of forcing terms in a space of periodic functions with zero mean value. The approach is based on global variational methods.

Key words: Heteroclinic solutions; Variational Methods; Implicit Function Theorem.

Riassunto. — *Densità delle dinamiche caotiche in equazioni di tipo pendolo forzato.* Si annuncia che una classe di problemi contenente l'equazione del pendolo forzato periodicamente presenta le principali caratteristiche della dinamica caotica per un insieme denso di termini forzanti nell'insieme delle funzioni periodiche a media nulla. I metodi sono di natura variazionale.

1. Introduction

Consider the equation

$$\ddot{u}(t) + g(u(t)) = h(t); \tag{1}$$

we assume that the functions h and $G(x) := \int_0^x g(s)\,ds$ satisfy the following hypotheses:

(H1) $h \in \mathcal{C}(\mathbb{R};\mathbb{R})$ is n-periodic, $n \in \mathbb{N}$, and $\int_0^n h(t)\,dt = 0$;

(H2) $G \in \mathcal{C}^3(\mathbb{R};\mathbb{R})$ is S-periodic;

(H3) if $0 < |x-y| < S$ and $g(x) = g(y) = 0$, then $G(x) \neq G(y)$.

When $g(x) = \sin x$, equation (1) becomes the classical forced pendulum equation, which is the main motivation for the results described in the present paper. Clearly, in this case, assumptions (H2)-(H3) are satisfied.

Let

$$X = \bigcup_{n\in\mathbb{N}} X_n = \bigcup_{n\in\mathbb{N}} \left\{ h \in \mathcal{C}(\mathbb{R};\mathbb{R}) \;/\; h \text{ is } n\text{-periodic and } \int_0^n h(t)\,dt = 0 \right\}.$$

The object of this *Note* is to announce that the set of forcing terms $h \in X$ giving rise to chaotic dynamics in equation (1) is dense in X. Since there is no precise nor unique definition of chaotic dynamics, we will use the following agreement.

Definition 1.1. *We say that equation* (1) *displays chaotic dynamics if*

i) *the solutions of* (1) *depend sensitively on the initial conditions*;

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ii) *equation* (1) *has infinitely many periodic solutions with diverging periods*;
iii) *equation* (1) *has an uncountable number of bounded, nonperiodic solutions*;
iv) *the Poincaré map associated to equation* (1) *has positive topological entropy*.

We refer to [16] for comments on the validity of *i*), ... , *iv*) above as indicators of the chaotic behavior of the dynamics of an equation of the form (1). Our main result is the following.

THEOREM 1.2. *Assume* (H2)-(H3) *hold. Then the set of forcing terms* $h \in X$ *for which equation* (1) *displays chaotic dynamics is dense in* X.

In order to prove Theorem 1.2 we use global variational techniques, aimed at the construction of connecting orbits. A similar procedure has been used by Mather in [9] in the discrete case. Genericity results for chaotic dynamics with global variational techniques have been obtained recently in [2], where the authors studied problems modelled on the Duffing equation.

Our approach takes advantage of results obtained in a recent paper, [4] (see also [6, 13]), where the authors proved that the existence of a certain class of multibump type heteroclinic solutions to periodic motions implies the requirements of Definition 1.1. The main result of [4] (see Theorem 2.1 below) states the conditions to obtain the desired class of multibump solutions. In the present work we show that the assumptions of Theorem 2.1 are satisfied for h in a dense subset of X.

The proof can be subdivided in three steps. First we make use of a result in the spirit of the Sard-Smale theorem to simplify the setting of the problem; next we show that a certain set of heteroclinic solutions enjoys some regularity properties and finally we use the regularity to compute an analogue of the Poincaré function (a primitive of the Melnikov function) and we derive from it the required conclusion. A part of the last step is reminiscent of the recent work [3], though our setting is more complex. The details of the proofs are contained in the forthcoming paper [5].

NOTATION. If u is an n-periodic function, we denote by $\overline{u}$ its mean value $\frac{1}{n}\int_0^n u(t)\,dt$ and by $\widetilde{u} = u - \overline{u}$ its zero mean part. By $[u_0, u_1]$, when u_0 and u_1 are periodic (continuous) functions, such that $u_0(t) \leq u_1(t)$ for all t, we denote the order interval $\{u : \mathrm{dom}(u) \to \mathbb{R} \;/\; u_0(t) \leq u(t) \leq u_1(t) \;\; \forall\, t \in \mathrm{dom}(u)\}$. Lastly, H^1_{loc} stands for $H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$.

2. PRELIMINARY RESULTS

We now state the main Theorem of [4]. To start with, let $h \in X_n$, denote by $L(u) = \frac{1}{2}\dot{u}^2 - G(u) + h(t)u$ the Lagrangian associated to (1) and let f be the action functional

$$f(u) = \int_0^n L(u)\,dt$$

defined over $E_n = \{u \in H^1_{\mathrm{loc}} \;/\; u(t+n) = u(t) \;a.e.\}$. It is well known (see for example [14]) that under assumptions (H1) and (H2) only, equation (1) possesses an ordered family of n-periodic solutions, which are the global minimizers of f over E_n.

Assume now that u_0 and u_1 are two *consecutive* minimizers, in the sense that there are no other global minimizers of f lying in the order interval $[u_0\,,\,u_1]$. Set

$$c_h = f(u_0) = f(u_1) = \min_{E_n} f.$$

The main result of [4] is based on the construction of heteroclinic and homoclinic solutions to u_0 and u_1 when t tends to $\pm\infty$. To simplify notation we will write $q(-\infty) = u_0$ instead of $\lim_{t\to-\infty}(q(t) - u_0(t)) = 0$ for example, and similar expressions at $+\infty$. Let

$$\Gamma(u_0\,,\,u_1) = \{q \in H^1_{\rm loc} \;/\; q(-\infty) = u_0\,,\; q(+\infty) = u_1\} \tag{2}$$

and

$$\Gamma(u_1\,,\,u_0) = \{q \in H^1_{\rm loc} \;/\; q(-\infty) = u_1\,,\; q(+\infty) = u_0\}\,, \tag{3}$$

and consider the functional $J : \mathrm{dom}(J) \subset H^1_{\rm loc} \to \mathbb{R}$ defined by

$$J(q) = \sum_{j\in\mathbb{Z}} \left(\int_{jn}^{(j+1)n} L(q)\,dt - c_h \right).$$

The *renormalized* functional J has been introduced by P.H. Rabinowitz in [10] and used to prove existence and multiplicity of heteroclinic and homoclinic solutions to periodic motions in [10-1]. To complete our description, let

$$\mathcal{S}(u_0\,,\,u_1) = \left\{ q(0) \in (u_0(0)\,,\,u_1(0)) \;/\; q \in \Gamma(u_0\,,\,u_1)\,,\; J(q) = \min_{\Gamma(u_0\,,\,u_1)} J \right\}$$

and

$$\mathcal{S}(u_1\,,\,u_0) = \left\{ q(0) \in (u_0(0)\,,\,u_1(0)) \;/\; q \in \Gamma(u_1\,,\,u_0)\,,\; J(q) = \min_{\Gamma(u_1\,,\,u_0)} J \right\}.$$

These are the sets of points in the interval $\mathcal{I}(u_0\,,\,u_1) := (u_0(0)\,,\,u_1(0))$ through which there passes a minimal level heteroclinic. Notice that, by the results of [4], they are infinite subsets of $\mathcal{I}(u_0\,,\,u_1)$, and also that if the problem is autonomous, then they both coincide with $\mathcal{I}(u_0\,,\,u_1)$, by invariance under real time shifts.

We can now state the main result from [4]. For further use we rephrase it here in the following form.

THEOREM 2.1 [4]. *Assume that* (H1)-(H2) *hold. If f has two consecutive minimizers u_0 and u_1 over E_n and*

$$\mathcal{S}(u_0\,,\,u_1) \neq \mathcal{I}(u_0\,,\,u_1) \qquad \textit{and} \qquad \mathcal{S}(u_1\,,\,u_0) \neq \mathcal{I}(u_0\,,\,u_1)\,, \tag{$*$}$$

then equation (1) *displays chaotic dynamics.*

In the more classical setting, where u_0 and u_1 are assumed to be hyperbolic, it has been proved in [4] that $(*)$ is equivalent to $W^s(u_0) \neq W^u(u_1)$ and $W^u(u_0) \neq W^s(u_1)$, a weaker condition than the transversality of the intersection of the stable and unstable manifolds relative to u_0 and u_1 (see also [15]).

In order to prove that equation (1) displays chaotic dynamics for a dense set (in X) of forcing terms we now show that there is such a set where the assumptions of Theorem 2.1 hold.

3. Scheme of the proof

The proof is divided in three steps, as described in the Introduction. First of all we show that equation (1) has consecutive minimizers of a special kind for a dense subset of forcing terms $h \in X$. We begin with the following statement. From now on we assume, without further repetition, that assumptions (H2)-(H3) hold.

Proposition 3.1. *For all* $n \in \mathbb{N}$ *the set* $\widehat{X}_n$ *of forcing terms* $h \in X_n$ *for which the functional* f *has only nondegenerate critical points over* E_n *is open and dense in* X_n.

The proof of this result can be found in [8], under the assumption that the set of zeroes of g be totally disconnected. In our case it is enough to notice that (H3) implies this property on g. We therefore assume, from now on, that h in equation (1) is chosen in $\widehat{X}_n$. For if this is not the case, we can modify it as little as we wish and obtain the desired property.

By Proposition 3.1, f has, up to translations by integer multiples of the space period S, only a finite number of critical points, and hence of global minimizers. Moreover, using (H3) and the Hahn-Banach Theorem, we can prove the following result.

Proposition 3.2. *Let* $h \in \widehat{X}_n$. *There exists* $h_0 \in X_n$ *such that for all* $\delta > 0$ *small enough, the functional* $f_\delta : E_n \to \mathbb{R}$ *defined by*

$$f_\delta(u) = \int_0^n L(u)\,dt + \delta \int_0^n h_0 u\,dt$$

has only one global minimizer (*up to translations by integer multiples of* S), *which is still nondegenerate.*

This ends the first step. Now choose $\delta > 0$ so small that Proposition 3.2 holds, fix $k \in \mathbb{N}$ and $h_1 \in X_{kn}$ and consider the equation

$$\ddot{u} + g(u) = h + \delta h_0 + \varepsilon h_1. \tag{4}$$

By the preceding results it is not difficult to show that for every $\varepsilon \in \mathbb{R}$ small enough, equation (4) has only one solution with mean value in $[0\,, S)$ which minimizes the corresponding action functional. We call u_0^ε this solution and we set $u_1^\varepsilon = u_0^\varepsilon + S$. Then u_0^ε and u_1^ε are consecutive minimizers, and we can try to apply Theorem 2.1. This will work if we are able to show that assumption $(*)$ holds for some choice of ε. We analyze only the first part of condition $(*)$, the other one being handled in a similar way.

Let J^ε be the functional J with h replaced by $h + \delta h_0 + \varepsilon h_1$.

Arguing indirectly, assume that $\mathcal{S}(u_0^\varepsilon\,, u_1^\varepsilon) = \mathcal{I}(u_0^\varepsilon\,, u_1^\varepsilon)$ for all ε small, say for all $\varepsilon \in (-\bar{\varepsilon}\,, \bar{\varepsilon})$. This means (see [4]) that for all $\varepsilon \in (-\bar{\varepsilon}\,, \bar{\varepsilon})$ and all $x \in \mathcal{I}(u_0^\varepsilon\,, u_1^\varepsilon)$, there exists a unique q_x^ε such that

$$q_x^\varepsilon(0) = x \qquad \text{and} \qquad J^\varepsilon(q_x^\varepsilon) = \min_{\Gamma(u_0^\varepsilon, u_1^\varepsilon)} J^\varepsilon.$$

The most technical result concerning q_x^ε is contained in the next proposition. To state it, let $\sigma : \mathbb{R} \to [0, S]$ be a smooth function such that $\sigma(-\infty) = 0, \sigma(+\infty) = S$ and $\dot\sigma \in H^1(\mathbb{R};\mathbb{R})$. Then we can write q_x^ε as

$$q_x^\varepsilon = u_0^\varepsilon + \sigma + \psi_x^\varepsilon,$$

with $\psi_x^\varepsilon \in H^2(\mathbb{R};\mathbb{R})$.

PROPOSITION 3.3. *In the above assumptions,*

$$\varepsilon \mapsto u_0^\varepsilon \text{ is } C^1 \text{ from } (-\bar\varepsilon, \bar\varepsilon) \text{ to } E_{kn},$$

$$(x, \varepsilon) \mapsto \psi_x^\varepsilon \text{ is } C^1 \text{ from } \mathcal{I}(u_0^\varepsilon, u_1^\varepsilon) \times (-\bar\varepsilon, \bar\varepsilon) \text{ to } H^2(\mathbb{R};\mathbb{R}).$$

The proof of this proposition (actually of the second part) is rather involved and uses a Lyapunov-Schmidt reduction. Moreover, also after the reduction the proof is not trivial since it involves an operator which is not of the form «Identity + Compact». We omit the technical details, beyond the scope of this note.

The regularity of q_x^ε allows us to arrive at the following conclusion. This is the last step of the proof.

PROPOSITION 3.4. *In the above assumptions, if $h_1 \in X_{kn}$ is such that $\mathcal{S}(u_0^\varepsilon, u_1^\varepsilon) = \mathcal{I}(u_0^\varepsilon, u_1^\varepsilon)$ for all ε in a neighborhood of zero, then for all $x, y \in \mathcal{I}(u_0, u_1)$,*

$$\int_{-\infty}^{+\infty} (q_x^0 - q_y^0) h_1 \, dt = 0.$$

SKETCH OF THE PROOF. Take $x, y \in \mathcal{I}(u_0, u_1)$ such that $x, y \in \mathcal{I}(u_0^\varepsilon, u_1^\varepsilon)$ for every small ε. Then for all such ε's,

$$J^\varepsilon(q_x^\varepsilon) - J^\varepsilon(q_y^\varepsilon) = 0. \tag{5}$$

The regularity proved in Proposition 3.3 and the fact that each q_z^ε tends to u_0^ε and to u_1^ε exponentially at $-\infty$ and $+\infty$ respectively (recall that each u_0^ε is hyperbolic), allows us to differentiate (5) with respect to ε. The fact that each q_z^ε solves (4) and some computing show that

$$\frac{d}{d\varepsilon}\left(J^\varepsilon(q_x^\varepsilon) - J^\varepsilon(q_y^\varepsilon)\right)\Big|_{\varepsilon=0} = \int_{-\infty}^{+\infty} (q_x^0 - q_y^0) h_1 \, dt,$$

which gives the desired equality. $\square$

This computation can be repeated for all $k \in \mathbb{N}$ and all $h_1 \in X_{kn}$ for which $\mathcal{S}(u_0^\varepsilon, u_1^\varepsilon) = \mathcal{I}(u_0^\varepsilon, u_1^\varepsilon)$ in a neighborhood of zero; notice that q_x^0 and q_y^0 do not depend on ε. In this way we see that if the first part of $(*)$ does not hold for any $h_1 \in X_{kn}$

with $k \in \mathbb{N}$, then

$$\int_{-\infty}^{+\infty} (q_x^0 - q_y^0) h_1 \, dt = 0 \quad \text{for all} \quad h_1 \in \bigcup_{k \in \mathbb{N}} X_{kn} \,,$$

which readily shows that $q_x^0 - q_y^0 \equiv 0$. This is a contradiction if $x \neq y$.

We can therefore find (for infinitely many $k \in \mathbb{N}$) an $h_1 \in X_k$ and a number ε as small as we please such that the first part of $(*)$ is verified for equation (4). By construction the set of h_1's enjoying this property is open in each X_{kn}; then, repeating the above argument, we can perturb these functions as little as we like, until we obtain that, when k is large enough, also the second part of $(*)$ holds.

We have thus shown that as close as we wish to $h \in X_n$ there is some function $\widehat{h} \in X$ for which equation

$$\ddot{u}(t) + g(u(t)) = \widehat{h}(t)$$

satisfies all the assumptions of Theorem 2.1, and therefore displays chaotic dynamics. Since the above reasoning does not depend on the minimal period of the starting forcing term h in X, this readily proves Theorem 1.2.

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