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A Note on height pairings on polarized abelian varieties

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Geometria. — *A* Note *on height pairings on polarized abelian varieties.* Nota di Valerio Talamanca, presentata (*) dal Corrisp. E. Arbarello.

Abstract. — Let A be an abelian variety defined over a number field k. In this short *Note* we give a characterization of the endomorphisms that preserve the height pairing associated to a polarization. We also give a functorial interpretation of this result.

Key words: Abelian varieties; Height pairings; Endomorphisms.

Riassunto. — *Una* Nota *sugli accoppiamenti associati alle altezze sulle varietà abeliane polarizzate.* Sia A una varietà abeliana definita su un campo di numeri k. In questa breve *Nota* diamo una caratterizzazione degli endomorfismi che lasciano invariata la forma bilineare associata all'altezza canonica definita a partire da una polarizzazione. Diamo, inoltre, un'interpretazione funtoriale di questo risultato.

1. Introduction

Let A be an abelian variety defined over a number field k. In [4], A. Néron introduced the canonical height pairing: $\langle\ ,\ \rangle : \widehat{A}(\overline{k}) \times A(\overline{k}) \longrightarrow \mathbb{R}$, between A and its dual abelian variety, $\widehat{A}$. This pairing satisfies the following fundamental properties:

HP1 The pairing $\langle\ ,\ \rangle$ is bilinear.
HP2 If $f : A \to B$ is a k-homomorphism, then

$$\langle \widehat{f}(b), a\rangle = \langle b, f(a)\rangle$$

for all $a \in A(\overline{k})$ and $b \in \widehat{B}(\overline{k})$.

A *polarization* on A is an isogeny $\lambda : A \to \widehat{A}$, such that $\lambda_{\overline{k}} = \varphi_{\mathcal{L}}$ for some ample invertible sheaf on $A_{\overline{k}} = A \times_{\mathrm{spec}\, k} \mathrm{spec}\, \overline{k}$ where $\varphi_{\mathcal{L}}(a) = t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}$, and t_a is the translation by a map (cf. [3]). To any polarization λ we can then associate a symmetric bilinear pairing

$$\begin{aligned} \langle\ ,\ \rangle_\lambda : A(\overline{k}) \times A(\overline{k}) &\longrightarrow \mathbb{R}, \\ (a, b) \quad &\longmapsto \langle \lambda(a), b\rangle, \end{aligned}$$

which is called the *height pairing associated to* λ.

In this short *Note* we give a characterization of the endomorphisms that preserve the height pairing associated to a polarization (actually, we prove a slightly more general result see the proposition below). In order to state our functorial interpretation of this result we need to define the category $\mathfrak{hgm}$, of height Galois modules. This is done as follows:

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objects: pairs $(G, \langle\ ,\ \rangle_G)$, where G is an abelian group endowed with an action of $\Gamma = \mathrm{Gal}\left(\overline{k}/k\right)$; $\langle\ ,\ \rangle_G$ is a symmetric bilinear real valued pairing on G, which is Γ-equivariant if we let Γ act trivially on $\mathbb{R}$.

morphisms: Γ-equivariant homomorphisms $f : G \to H$ such that

$$(*) \qquad \langle f(a), f(a')\rangle_H = \langle a, a'\rangle_G$$

for all $a\, a' \in G$.

Theorem. *Let k be a number field and $\Gamma = \mathrm{Gal}\left(\overline{k}/k\right)$. Let $\mathfrak{pav}_k$ denote the category of polarized abelian varieties defined over k. Then, the functor $\mathcal{F} : \mathfrak{pav}_k \to \mathfrak{hgm}$, which assign to (A, λ) the height Galois module $\mathcal{F}(A, \lambda) = (A(\overline{k}), \langle\ ,\ \rangle_\lambda)$, and to any morphism $f : A \longrightarrow B$ the induced morphism $f : A(\overline{k}) \longrightarrow B(\overline{k})$, is fully faithful.*

2. Height pairings and homomorphisms

Let $(A, \lambda), (B, \eta)$, be two polarized abelian varieties defined over k. An homomorphism of polarized abelian varieties is an homomorphism $f : A \longrightarrow B$ such that $\lambda = \widehat{f} \circ \eta \circ f$. We denote by $\mathrm{Hom}_k\big((A, \lambda), (B, \eta)\big)$ the set formed by those homomorphisms from (A, λ) to (B, η) that are defined over k.

Our aim is the following:

Proposition. *Let $g : A \to B$ be a morphism defined over k. Suppose that λ, η are polarizations on A and B respectively. Let $g = t_u \circ f$, where $u \in B(k)$, and $f : A \longrightarrow B$ is a homomorphism. Then*

$$(2.1) \qquad \langle g(a), g(a')\rangle_\eta = \langle a, a'\rangle_\lambda \qquad \forall\, a, a' \in A(\overline{k})$$

if and only if $f \in \mathrm{Hom}_k\big((A, \lambda), (B, \eta)\big)$ and u is a torsion point.

Corollary. *Let (A, λ) be a polarized abelian variety and f an endomorphism of A. Then*

$$\langle f(a), f(a')\rangle_\lambda = \langle a, a'\rangle_\lambda \qquad \forall\, a, a' \in A(\overline{k})$$

if and only if $\lambda = \widehat{f} \circ \lambda \circ f$.

We need a preliminary lemma.

Lemma. *Let λ and η be two polarizations on A. Then $\langle\ ,\ \rangle_\lambda = \langle\ ,\ \rangle_\eta \iff \lambda = \eta$.*

Proof. Let λ and η be two polarizations on A such that $\langle\ ,\ \rangle_\lambda = \langle\ ,\ \rangle_\eta$. Then

$$\langle \lambda(a) - \eta(a), a'\rangle = \langle a, a'\rangle_\lambda - \langle a, a'\rangle_\eta = 0$$

for all a' in A. Since the kernel on each side of the Néron pairing is the torsion subgroup of $A(\overline{k})$ (see, *e.g.* [2, Theorem 5.6.3]) we find that for every $a \in A(\overline{k})$ there exists $n \in \mathbb{Z}$, depending on a, such that $[n]\lambda(a) = [n]\eta(a)$. Let C be a simple abelian subvariety of A. Then $\lambda(a) = \eta(a)$ for infinitely many $a \in C(\overline{k})$, and thus λ and η coincide when restricted to C. The Poincaré reducibility theorem yields the lemma. □

PROOF OF THE PROPOSITION. We start by proving the proposition for homomorphisms. If a, $a' \in A(\overline{k})$, then

$$(2.2) \qquad \langle f(a), f(a')\rangle_\eta = \langle \eta(f(a)), f(a')\rangle = \langle (\widehat{f} \circ \eta \circ f)(a), a'\rangle = \langle a, a'\rangle_{\widehat{f}\circ\eta\circ f}.$$

If $f \in \mathrm{Hom}((A, \lambda), (B, \eta))$, then $\widehat{f} \circ \eta \circ f = \lambda$, and hence f has the desired property. Conversely, suppose that $\langle f(a), f(a')\rangle_\eta = \langle a, a'\rangle_\lambda$ for all $a, a' \in A(\overline{k})$. Then, by (2.2), we have $\langle\ ,\ \rangle_{\widehat{f}\circ\eta\circ f} = \langle\ ,\ \rangle_\lambda$. Therefore, $\widehat{f} \circ \eta \circ f = \lambda$ by the above lemma. Now we deal with the general case. Suppose $g = t_u \circ f$, where $f \in \mathrm{Hom}((A, \lambda), (B, \lambda))$, and $u \in B(k)$ a torsion point. The bilinearity of $\langle g(a), g(b)\rangle_\eta$ combined with (2.2), gives

$$\langle g(a), g(b)\rangle_\eta = \langle f(a), f(a')\rangle_\eta = \langle a, a'\rangle_\lambda.$$

Finally, suppose that $g = t_u \circ f$ satisfies (2.1). Then $\langle g(a), u\rangle_\eta = 0$ for all $a \in A(\overline{k})$. Using the bilinearity of $\langle\ ,\ \rangle_\eta$ and (2.2), we find

$$\langle a, a'\rangle_\lambda = \langle f(a), f(a')\rangle_\eta = \langle a, a'\rangle_{\widehat{f}\circ\eta\circ f}.$$

It then follows from the lemma above that $\widehat{f} \circ \eta \circ f = \lambda$. It remains to show that u is a torsion point. Let $\mathcal{L}$ be an ample invertible sheaf on such that $\eta_{\overline{k}} = \varphi_{\mathcal{L}}$. Then $\mathcal{L}_0 = \mathcal{L} \otimes \mathcal{L}^-$ (where $\mathcal{L}^- = [-1]^*\mathcal{L}$) is ample and symmetric. Since $\mathcal{L} \otimes (\mathcal{L}^-)^{-1}$ is algebraically equivalent to zero, we have that $\varphi_{\mathcal{L}} = \varphi_{\mathcal{L}^-}$, so

$$\langle u, u\rangle_{\varphi_{\mathcal{L}_0}} = 2\langle u, u\rangle_{\varphi_{\mathcal{L}}} = 2\langle u, u\rangle_\eta = \langle 0, 0\rangle_\lambda = 0.$$

But, $\mathcal{L}_0$ being symmetric, $\langle u, u\rangle_{\varphi_{\mathcal{L}_0}}$ is proportional to the canonical height associated to $\mathcal{L}_0$, which, for an ample symmetric divisor, vanishes only on torsion points. □

A FUNCTORIAL INTERPRETATION

The above proposition has a functorial interpretation as we shall now show. Let $\mathfrak{pav}_k$ denote the category whose objects are polarized abelian varieties defined over k, and let $\mathfrak{hgm}$ be the category of height Galois modules, which we defined in the Introduction. Given $(G, \langle\ ,\ \rangle_G)$ and $(H, \langle\ ,\ \rangle_H)$, we denote by $\mathrm{Hom}_{\mathfrak{h}}(G, H)$ the set of morphisms (in the category of height Galois modules) from $(G, \langle\ ,\ \rangle_G)$ to $(H, \langle\ ,\ \rangle_H)$. We define a functor $\mathcal{F} : \mathfrak{pav}_k \to \mathfrak{hgm}$ as follows: given a polarized abelian variety (A, λ) defined over k we let $\mathcal{F}(A, \lambda) = (A(\overline{k}), \langle\ ,\ \rangle_\lambda)$. Given a k-morphism $f : (A, \lambda) \to (B, \eta)$ then $\mathcal{F}(f)$ is just the restriction of f to $A(\overline{k})$, which is Γ-equivariant because f is defined over k. Moreover $\mathcal{F}(f)$ satisfies $(*)$ by the above proposition. The only thing that remains to be verified is that $(A(\overline{k}), \langle\ ,\ \rangle_\lambda)$ is an object of $\mathfrak{hgm}$, *i.e.* that $\langle\ ,\ \rangle_\lambda$ is invariant under the action of Γ. Recall that the canonical height pairing coincides with the canonical height associated to the Poincaré bundle on $\widehat{A} \times A$ (this can be seen by comparing [4, Section 14] and [5, Section 3.4], also cf. [7, Appendix A1, Proposition 6]). But the Poincaré bundle is defined over the ground field, and absolute projective heights are invariant under the action of Γ (see [6, Lemma 5.10]).

Therefore, $\langle\ ,\ \rangle$ is invariant under the action of Γ. Thus

$$\langle \sigma(a)\,,\,\sigma(a')\rangle_\lambda = \langle \lambda\big(\sigma(a)\big)\,,\,\big(\sigma(a')\big)\rangle = \langle \sigma\big(\lambda(a)\big)\,,\,(\sigma(a'))\rangle = \langle \lambda(a)\,,\,a'\rangle = \langle a\,,\,a'\rangle_\lambda .$$

where we used that λ is defined over k.

Theorem. *Let k be a number field. Then the functor $\mathcal{F} : \mathfrak{pav}_k \longrightarrow \mathfrak{hgm}$ is fully faithful.*

Proof. The faithfulness of $\mathcal{F}$ follows directly from the above proposition. To prove that $\mathcal{F}$ is full let A and B be two abelian varieties defined over k, and suppose that $\widetilde{f}$ is in $\mathrm{Hom}_{\mathfrak{h}}\big(A(\overline{k})\,,\,B(\overline{k})\big)$. In particular $\widetilde{f}$ is a Γ-equivariant homomorphism from $A(\overline{k})$ to $B(\overline{k})$. It was shown by Faltings (as a non-trivial consequence of the Tate's conjecture) that the natural injection $\mathrm{End}_k(A) \hookrightarrow \mathrm{End}_\Gamma\big(A(\overline{k})\big)$ is an isomorphism (see [1, Theorem 5, p. 205]). Applying this result to $A \times B$, we find that there exists an f belonging to $\mathrm{Hom}_k(A,\, B)$ such that $\mathcal{F} = \widetilde{f}$. By assumption $\widetilde{f}$ belongs to $\mathrm{Hom}_{\mathfrak{h}}\big(A(\overline{k})\,,\,B(\overline{k})\big)$, thus $\langle a,\, a'\rangle_\lambda = \langle f(a)\,,\,f(a')\rangle_\eta$ for all $a\, a' \in A(\overline{k})$. By the above proposition f belongs to $\mathrm{Hom}_k\big((A,\,\lambda)\,,\,(B,\,\eta)\big)$. □

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