Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Levi's forms of higher codimensional submanifolds

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **2** (1991), n.1, p. 29–33.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1991_9_2_1_29_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1991.

Rend. Mat. Acc. Lincei
s. 9, v. 2:29-33 (1991)

Equazioni a derivate parziali. — *Levi's forms of higher codimensional submanifolds.* Nota di ANDREA D'AGNOLO e GIUSEPPE ZAMPIERI, presentata (*) dal Socio G. SCORZA DRAGONI.

ABSTRACT. — Let $X \cong \mathbf{C}^n$, let M be a C^2 hypersurface of X, S be a C^2 submanifold of M. Denote by L_M the Levi form of M at $z_0 \in S$. In a previous paper [3] two numbers $s^{\pm}(S,p)$, $p \in (\dot{T}^*_S X)_{z_0}$ are defined; for $S = M$ they are the numbers of positive and negative eigenvalues for L_M. For $S \subset M$, $p \in S \times_M \dot{T}^*_M X$, we show here that $s^{\pm}(S,p)$ are still the numbers of positive and negative eigenvalues for L_M when restricted to $T^{\mathbf{C}}_{z_0} S$. Applications to the concentration in degree for microfunctions at the boundary are given.

KEY WORDS: Partial differential equations on manifolds; Boundary value problems; Theory of functions.

RIASSUNTO. — *Forme di Levi a sottovarietà di codimensione superiore.* Sia $X \cong \mathbf{C}^n$, M una ipersuperficie di classe C^2 di X, S una sottovarietà C^2 di M. Sia L_M la forma di Levi di M al punto $z_0 \in S$. In un precedente lavoro [3] si definiscono dei numeri $s^{\pm}(S,p)$, $p \in (\dot{T}^*_S X)_{z_0}$ che per $S = M$ coincidono con i numeri di autovalori positivi e negativi di L_M. Per $S \subset M$, $p \in S \times_M \dot{T}^*_M X$, si prova che $s^{\pm}(S,p)$ sono ancora i numeri di autovalori positivi e negativi di L_M ristretta a $T^{\mathbf{C}}_{z_0} S$. Se ne dà applicazione alla concentrazione in grado di microfunzioni al bordo.

1. Let X be a complex analytic manifold of dimension n. We denote by $\tau: TX \to X$ the tangent bundle and by $\pi: T^*X \to X$ the cotangent bundle to X. If $X^{\mathbf{R}}$ denotes the underlying real analytic manifold structure on X, we recall that there is a natural identification $T^*(X^{\mathbf{R}}) \cong (T^*X)^{\mathbf{R}}$. We will denote by ∂ the holomorphic differential on X, and by $d = \partial + \bar{\partial}$ the differential on $X^{\mathbf{R}}$.

Let M be a C^2 hypersurface of X and S be a C^2 submanifold of M of real codimension $s - 1$. We denote by $T^*_S X$ the conormal bundle to S in X, a closed submanifold of $T^* X^{\mathbf{R}}$.

Take a point $z_0 \in S$ and assume that, locally at z_0, one may express M as $\{z \in X;\ \phi(z) = 0\}$ and S as the set of zeros for the functions $\phi_i\ (i = 1, \ldots, s)$. Here ϕ and the ϕ_i are real valued C^2 functions on X. Let $p = d\,\phi(z_0) \in S \times_M \dot{T}^*_M X \subset \dot{T}^*_S X$. Let L_M be the Levi form of M at z_0. Recall that, in a local system of coordinates $(z) \in X$ at z_0, one has $L_M = (\partial^2 \phi(z_0)/\partial z_i \partial \bar{z}_j)_{i,j}$.

Let us set

$$T_{z_0} S = \{z \in T_{z_0} X;\ Re\langle z, \partial\phi_i\rangle = 0\ \forall i\}, \qquad T^{\mathbf{C}}_{z_0} S = \{z \in T_{z_0} X;\ \langle z, \partial\phi_i\rangle = 0\ \forall i\},$$

$$\lambda_S(p) = T_p T^*_S X, \qquad \lambda_0(p) = T_p \pi^{-1}\pi(p),$$

$$\gamma(S,p) = \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)), \qquad s(S,p) = \tau(\lambda_S(p), i\lambda_S(p), \lambda_0(p))/2,$$

$$r(S,p) = n - \mathrm{codim}_X S + 2\gamma(S,p) - \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p)),$$

(*) Nella seduta del 14 giugno 1990.

here $\tau(\cdot,\cdot,\cdot)$ denotes the Maslov index for three Lagrangean planes and $\mathrm{codim}_X S$ is the real codimension of S in X (cf. [3, §7]).

PROPOSITION 1.1. *On has* $s(S,p) = \mathrm{sgn}(L_M|_{T^{C}_{z_0}S})$, *where* sgn *denotes the signature.*

PROOF. The proof goes as in [2, Prop. 11.2.7] so we point out only the main lines.

Define the map:

$$\psi: S \to T^*_S X, \qquad z \mapsto \partial\phi(z).$$

Let $\psi_*: \boldsymbol{C}\otimes_{\boldsymbol{R}} T_{z_0}S \to \lambda_S(p) + i\lambda_S(p) \subset T_p T^*X$, be the map induced by ψ on the complexification $\boldsymbol{C}\otimes_{\boldsymbol{R}} T_{z_0}S$ of $T_{z_0}S$. If we identify $\boldsymbol{C}\otimes_{\boldsymbol{R}} T_{z_0}S$ to the subset of $T_{z_0}X \oplus \overline{T_{z_0}X}$ defined by $\{(v,w);\ \langle v, \partial\phi_i\rangle + \langle w, \bar\partial\phi_i\rangle = 0,\ \forall i\}$, we have: $\psi_*(v,w) = (v; \zeta = \partial\langle\partial\phi, v\rangle + \partial\langle\bar\partial\phi, w\rangle)$. We have $\lambda_S(p) = \{(v,\zeta);\ Re\langle v, \partial\phi_i\rangle = 0,\ \zeta = \sum t_j \partial\phi_j(z_0) + \partial\langle\partial\phi, v\rangle + \partial\langle\bar\partial\phi, \bar v\rangle\}$.

We need now a lemma.

LEMMA 1.2. *One has* $\psi_* \overline{T_{z_0}S} + (\lambda_0(p)\cap\lambda_S(p))^{C} = \lambda_0(p)\cap(\lambda_S(p) + i\lambda_S(p))$.

PROOF. Recall that one has the identification $\psi_*\overline{T_{z_0}S} = \{(0,\zeta);\ \zeta = \partial\langle\bar\partial\phi, w\rangle, \langle w, \bar\partial\phi_i\rangle = 0,\ \forall i \leq s\}$, and also $(\lambda_0(p)\cap\lambda_S(p))^{C} = \{(0,\zeta); \zeta = \sum_i \tau_i \partial\phi_i(z_0), \tau_i \in \boldsymbol{C}\}$.

In conclusion:

$$\psi_*\overline{T_{z_0}S} + (\lambda_0(p)\cap\lambda_S(p))^{C} = \{(0,\zeta); \zeta = \sum_i \tau_i\partial\phi_i(z_0) + $$
$$+ \partial\langle\bar\partial\phi, w\rangle, \langle\bar\partial\phi_i w\rangle = 0\ \forall i \leq s\} = \lambda_0(p)\cap(\lambda_S(p) + i\lambda_S(p)). \qquad \text{Q.E.D.}$$

One sees that $\psi_*(0,v) + \psi_*(\bar v, 0) \in \lambda_S(p)$ and $\psi_*(0,v) - \psi_*(\bar v,0) \in i\lambda_S(p)$ and thus $\psi_*(\bar v, 0)$ is the «conjugate» to $\psi_*(0,v)$ in $T_pT^*X/(\lambda_S(p)\cap i\lambda_S(p))$ with respect to $\lambda_S(p)/(\lambda_S(p)\cap i\lambda_S(p))$. The conclusion is then as in [2]. Q.E.D.

PROPOSITION 1.3. *One has* $r(S,p) = \mathrm{rank}(L_M|_{T^{C}_{z_0}S})$.

PROOF. One has

$$\lambda_S(p)\cap i\lambda_S(p) = \{(v,\zeta); \langle v, \partial\phi_i\rangle = 0 \quad \forall i \leq s,\ \zeta = \sum_i t_i\partial\phi_i(z_0) + \partial\langle\partial\phi, v\rangle + \partial\langle\bar\partial\phi, v\rangle,$$
$$\zeta = \sum_i is_i\partial\phi_i(z_0) + \partial\langle\partial\phi, v\rangle - \partial\langle\bar\partial\phi, v\rangle\} =$$
$$= \{(v,\zeta); \langle v, \partial\phi_i\rangle = 0\ \forall i \leq s,\ \langle\partial\langle\bar\partial\phi, \bar v\rangle, w\rangle = 0,\ \forall w \in T^{C}_{z_0}S,$$
$$\zeta = \sum_i t_i\partial\phi_i(z_0) + \partial\langle\partial\phi, v\rangle + \partial\langle\bar\partial\phi, v\rangle,\ \sum_i(-t_i + is_i)\partial\phi_i(z_0) = 2\partial\langle\bar\partial\phi, \bar v\rangle\}.$$

Thus $\zeta - \partial\langle\partial\phi, v\rangle - \partial\langle\bar\partial\phi, v\rangle$ is the first term in a decomposition: $-2\partial\langle\bar\partial\phi, \bar v\rangle = \sum_i t_i\partial\phi_i(z_0) - i\sum_i s_i\partial\phi_i(z_0)$, t_i, $s_i \in \boldsymbol{R}$. This last decomposition being unique modulo $\lambda_S(p)\cap i\lambda_S(p)\cap\lambda_0(p)$, we get:

$$\dim_C(\lambda_S(p)\cap i\lambda_S(p)) = \dim_C\{v \in T^{C}_{z_0}S; \langle\partial\langle\bar\partial\phi, \bar v\rangle, w\rangle = 0,\ \forall w \in T^{C}_{z_0}S\} +$$
$$+ \dim_C(\lambda_S(p)\cap i\lambda_S(p)\cap\lambda_0(p)).$$

We have, recalling that $T^{C}_{z_0}S = T_{z_0}S \cap iT_{z_0}S$:

$$\dim_C (T^C_{z_0} S) = n - \operatorname{codim}_X S + \dim_C (\lambda_S (p) \cap i\lambda_S (p) \cap \lambda_0 (p)),$$

and hence:

$$\operatorname{rank} L_M |_{T^C_{z_0}S} = \dim_C (T^C_{z_0} S) - \dim_C \{v \in T^C_{z_0} S;\ \langle \partial \langle \bar\partial \phi, \bar v \rangle, w \rangle = 0,\ \forall w \in T^C_{z_0} S\} =$$
$$= (n - \operatorname{codim}_X S + \dim_C (\lambda_S (p) \cap i\lambda_S (p) \cap \lambda_0 (p))) -$$
$$-(\dim_C (\lambda_S (p) \cap i\lambda_S (p)) - \dim_C (\lambda_S (p) \cap i\lambda_S (p) \cap \lambda_0 (p))) = r(S,p). \qquad \text{Q.E.D.}$$

Let us define $s^{\pm}(S,p)$ as the solutions of the system:

$$\begin{cases} s(S,p) = s^+(S,p) - s^-(S,p) \\ r(S,p) = s^+(S,p) + s^-(S,p), \end{cases}$$

and $s^0(S,p)$ as: $s^0(S,p) = \dim_C (\lambda_S (p) \cap i\lambda_S (p)) - \gamma(S,p)$. Due to Propositions 1.1, 1.3, we get at once the following theorem.

THEOREM 1.4. *$s^+(S,p)$, $s^-(S,p)$ and $s^0(S,p)$ are the numbers of positive, negative and null eigenvalues for the form $L_M |_{T^C_{z_0}S}$.*

2. We give now application of the preceding results to the concentration in degree for microfunctions at the boundary.

We shall be working with the derived category $D^b(X)$ of the category of bounded complexes of sheaves of abelian groups on X. In [2] a bifunctor μhom: $D^b(X)^\circ \times D^b(X) \to D^b(T^*X)$ is defined. For a subset $Z \subset X$ one sets $\mu_Z(\mathcal{O}_X) = \mu hom(C_Z, \mathcal{O}_X)$, where $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X and C_Z is the sheaf which is 0 on $X \setminus Z$ and the constant sheaf with fiber $\mathbf{C}$ on Z.

In what follows π_M will denote the projection π_M: $T^*_M X \to M$, ω will denote the complex canonical 1-form on X, H the hamiltonian isomorphism $T^*T^*X \cong TT^*X$ induced by the symplectic 2-form $d\omega$ and, in the case of $T^*_M X$ being $d\omega^I$-symplectic, H^I will denote the hamiltonian isomorphism $T^*T^*_M X \cong TT^*_M X$ induced by the symplectic 2-form $d\omega^I$ (ω^I being the imaginary part of ω).

PROPOSITION 2.1. *Let $S \subset M \subset X$ be C^2 submanifolds of X, M being an hypersurface. Then, for $p \in S \times_M \dot T^*_M X$:*

(i) $0 \leqslant s^{\pm}(M,p) - s^{\pm}(S,p) \leqslant \operatorname{codim}_M S - [\gamma(S,p) - \gamma(M,p)]$,

(ii) $-\operatorname{codim}_M S + [\gamma(S,p) - \gamma(M,p)] \leqslant s^0(M,p) - s^0(S,p) \leqslant \operatorname{codim}_M S - [\gamma(S,p) - \gamma(M,p)]$.

PROOF. Recall that $T^C_{z_0}M = T_{z_0}M \cap iT_{z_0}M$. We have: $\dim_C T^C_{z_0}S = \dim_C T^C_{z_0}M - [\operatorname{codim}_M S - (\gamma(S,p) - \gamma(M,p))]$. Let us define (for $* = +, -, 0$):

$$V^*_M = \left\{ v \in T^C_{z_0}M;\ v \text{ and } v^\perp \text{ generate } T^C_{z_0}M,\ {}^t v L_M \bar v \begin{cases} >0 & \text{for } * = + \\ <0 & \text{for } * = - \\ =0 & \text{for } * = 0 \end{cases} \right\}$$

(where $v^\perp = \{w \in T^C_{z_0}M;\ {}^t v L_M \bar w = 0\}$), and similarly for V^*_S. One has that $s^*(M,p)$ is the maximal dimension for subspaces of V_M^*, and similarly for S. In order to prove that the right hand side estimates on (i) and (ii) hold, it is enough to observe that $V_S^* \subset V_M^* \cap T_{z_0}^C S$.

Let now $v_1, \ldots, v_s$ be a base for a subspace of V_S^+, completed in a basis $v_1, \ldots, v_s, \widetilde{v}_{s+1}, \ldots, \widetilde{v}_r$ of $T_{z_0}^C M$, and let $(a_{ij})_{i,j \leq r}$ be the matrix of L_M in such a base. If we write $v_{s+1} = \widetilde{v}_{s+1} - (a_{1s+1}/a_{11} v_1 + \ldots + a_{ss+1}/a_{ss} v_s)$, and similarly for v_j, $s+1 < j \leq r$, we get a new base for $T_{z_0}^C M$ such that ${}^t v_i L_M \bar{v}_j = 0$, $\forall i \leq s$, $\forall j \leq m$. This proves that the left hand side estimate on (i) holds.

In order to prove that the left hand side estimates on (ii) holds, assume at first that $\dim_C T_{z_0}^C M - \dim_C T_{z_0}^C S = 1$ and let $u \in T_{z_0}^C M \setminus T_{z_0}^C S$. Let v_1, $v_2 \in T_{z_0}^C S$, with ${}^t v_i (L_M|_{T_{z_0}^C S}) \bar{w} = 0$, $\forall i = 1,2$, $\forall w \in T_{z_0}^C S$. Assume that ${}^t v_1 L_M \bar{u} \neq 0$ and hence there exists an $\alpha \in C$ with ${}^t(\alpha v_1 + v_2) L_M \bar{u} = 0$. This means $(\alpha v_1 + v_2)^\perp = T_{z_0}^C M$. The case $\dim_C T_{z_0}^C M - \dim_C T_{z_0}^C S > 1$ is similarly proven. Q.E.D.

PROPOSITION 2.2. *Let $S \subset M \subset X$ be C^2 submanifolds of X, M a hypersurface, and take $p_0 \in S \times_M \dot{T}_M^* X$. Let $\Omega = M \setminus S$. Assume the following conditions for p in a neighborhood of p_0:*

(1) $\dim_R (\lambda_S(p) \cap \nu(p)) = 1$ *(here $\nu(p) = CH(\omega(p))$ is the radial direction),*

(2) $s^-(S,p) - \gamma(S,p)$ *is locally constant,*

(3) $s^-(M,p) - \gamma(M,p)$ *is locally constant,*

Then $\mu_\Omega(\mathcal{O}_X)$ is concentrated in degrees $[\alpha - 1, \alpha' \vee \alpha]$, where we set $\alpha = \operatorname{codim}_X M + s^-(M,p) - \gamma(M,p)$, $\alpha' = \alpha + \operatorname{codim}_M S - [\gamma(S,p) - \gamma(M,p)] - 1$.

PROOF. By [2, Prop. 2.3] we have that $\mu_M(\mathcal{O}_X)$ is concentrated in degree α and $\mu_S(\mathcal{O}_X)$ is concentrated in degree $\alpha'' = \operatorname{codim}_X S + s^-(S,p) - \gamma(S,p)$. Due to Corollary 2.1 we then have $\alpha \leq \alpha'' \leq \alpha' + 1$ and we conclude. Q.E.D.

REMARK 2.3. If we assume in addition that:

$$\begin{cases} T_M^* X \text{ is } d\omega^I \text{ symplectic} (\operatorname{codim}_X M = 1), \\ S \times_M T_M^* X \text{ is } d\omega^I \text{ involutive}, \\ ip_0 \notin T_S^* M \text{ in the identification } T_S^* M \cong iH^I(\pi_M^*(T_S^* M)) \end{cases}$$

then $\mu_\Omega(\mathcal{O}_X)$ is concentrated in degrees $[\alpha, \alpha + \operatorname{codim}_M S - 1]$. In fact $\mu_M(\mathcal{O}_X)$ is concentrated in degree $\alpha = \operatorname{codim}_X M + s^-(M,p)$. By a complex quantized contact transformation it is not restrictive to assume $s^-(M,p) = 0$, and hence $s^-(S,p) = 0$ due to Proposition 2.1. Moreover, since $(T_M^* X, d\omega^I)$ is symplectic and $S \times_M T_M^* X$ is regular involutive, then $\gamma(S,p) = 0$ (cf. [1]). This implies (cf. [3, Prop. 11.2.8]) that $\mu_S(\mathcal{O}_X)$ is concentrated in degree $\operatorname{codim}_X S + s^-(S,p)$. Note that if $\operatorname{codim}_X M + s^-(M,p) = \operatorname{codim}_X S + s^-(S,p)$, the concentration in degree α for $\mu_\Omega(\mathcal{O}_X)$ follows by applying a quantized contact transformation that interchanges $T_M^* X$ with $T_{M'}^* X'$ (for $M' \cong \boldsymbol{R}^n$) and $S \times_M T_M^* X$ with $S' \times_{M'} T_{M'}^* X'$. The concentration for $\mu_{\Omega'}(\mathcal{O}_X)(\Omega' = M' \setminus S')$ is then a classical argument due to P. Schapira.

REMARK 2.4. If we assume that:

$$\begin{cases} T^*_M X \text{ is } d\omega^I \text{ symplectic} (\operatorname{codim}_X M = 1), \\ \operatorname{codim}_M S = 1, \\ ip_0 \notin T^*_S M \text{ in the identification } T^*_S M \cong iH^I(\pi^*_M(T^*_S M)) \end{cases}$$

then $\mu_\Omega(\mathcal{O}_X)$ is concentrated in degree $\alpha = \operatorname{codim}_X M + s^-(M,p)$. In fact, by a quantized contact transformation, we can reduce to the case $s^-(S,p) \equiv 0$, $s^-(M,p) \equiv 0$, M being the boundary of a strictly pseudo-convex set, moreover, in this case of $\operatorname{codim}_S X = 2$, we have $\gamma(S,p) = 0$.

PROPOSITION 2.5. *Let* $S \subset M \subset X$ *be* C^2 *submanifolds of* X, *set* $\Omega = M \setminus S$ *and take* $p_0 \in S \times \dot{T}^*_M X$. *Then* $\mu_\Omega(\mathcal{O}_X)$ *is concentrated in degrees* $[\alpha - 1, \beta + \operatorname{codim}_M S - 1]$, *where we set* $\alpha = \operatorname{codim}_X M + s^-(M,p) - \gamma(M,p)$ *and* $\beta = n - s^+(M,p) + \gamma(M,p)$.

PROOF. Due to [3, Th. 2.2] one has that $\mu_M(\mathcal{O}_X)$ is concentrated in degree $[\alpha,\beta]$ and $\mu_S(\mathcal{O}_X)$ is concentrated in degree $[\alpha', \beta']$, for:

$$\alpha' = \operatorname{codim}_X S + s^-(S,p) - \gamma(S,p), \qquad \beta' = n - s^+(S,p) + \gamma(S,p).$$

Due to Corollary 2.1 we have

$$s^\pm(S,p) \geq s^\pm(M,p) - \operatorname{codim}_M S + [\gamma(S,p) - \gamma(M,p)]$$

and hence:

$$\alpha \leq \alpha',$$

$$\beta' \leq n - s^+(M,p) + \operatorname{codim}_M S - (\gamma(S,p) - \gamma(M,p)) + \gamma(S,p) = \beta + \operatorname{codim}_M S. \qquad \text{Q.E.D.}$$

REFERENCES

[1] A. D'AGNOLO - G. ZAMPIERI, *A vanishing theorem at the boundary for a class of systems with simple characteristics*. To appear.

[2] M. KASHIWARA - P. SCHAPIRA, *Microlocal study of sheaves*. Astérisque, 128, 1985.

[3] M. KASHIWARA - P. SCHAPIRA, *A vanishing theorem for a class of systems with simple characteristics*. Invent. Math., 82, 1985, 579-592.

[4] P. SCHAPIRA, *Condition de positivité dans une variété symplectique complexe. Applications à l'étude des microfonctions*. Ann. Sci. Ec. Norm. Sup., 14, 1981, 121-139.

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