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Multivalued nonpositone problems

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Analisi matematica. — *Multivalued non-positone problems.* Nota di DAVID ARCOYA e MARCO CALAHORRANO, presentata (*) dal Corrisp. A. AMBROSETTI.

ABSTRACT. — In this note, the existence of non-negative solutions for some multivalued non-positone elliptic problems is studied.

KEY WORDS: Elliptic multivalued problem; Discontinuous nonlinearities; Sub-linear and superlinear.

RIASSUNTO. — *Problemi di tipo «non-positone» a multivalori.* In questa nota si studia la esistenza di soluzioni non negative di certi problemi a multivalori ellittici non lineari.

0. INTRODUCTION

In this paper, we will consider the boundary value problem,

$$-\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $f\colon [0, +\infty) \to \mathbb{R}$ is a C^1-function with $f(0) < 0$ (non-positone).

Recently, Brown *et al.* [4] have proved a result of non-existence of non-negative radial solutions of (0.1), when Ω is a ball and f is a superlinear and increasing function. In concrete, it is proved there that, if $f = \lambda g$ with $\lambda \in \mathbb{R}$, then there exists $\lambda_0 > 0$ such that (0.1) has no such solutions for all $\lambda \geq \lambda_0$. For existence of at least one positive solution for λ sufficiently small, see [5].

Motivated by this result, we will study here the existence of non-negative solutions of the multivalued problem

$$-\Delta u(x) \in \bar{f}(u(x)) \ \textit{a.e.}\ \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad u \geq 0 \text{ in } \Omega \tag{0.2}$$

where $\bar{f}$ is the multivalued function defined by

$$\bar{f}(u) = \begin{cases} [f(0), 0], & \text{if } t = 0; \\ f(t), & \text{if } t > 0. \end{cases}$$

In contrast with [4], we will prove the existence of i) one non-zero C^1-solution of (0.2) if f is superlinear (with no further restrictions); ii) two non-zero and distinct solutions of (0.2) if f is asymptotically linear (not at resonance) verifying some additional condition.

There have been some works on elliptic problems with discontinuous nonlinearities where a suitable direct variational approach is used ([1], [6] and [10]). However, here we find more convenient (at least, in the superlinear case) to work on the approximating problems

$$-\Delta u = f_n(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{0.3}$$

(*) Nella seduta del 9 dicembre 1989.

where f_n is a sequence of functions which «converges» in some sense to f and $f_n(0)=0$. A convenient choice of f_n permits us to prove the existence of solutions of (0.3), which are necessarily positive. A simple limiting procedure allows us to obtain solutions of (0.2).

1. The sub-linear case

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary and $f\colon [0,+\infty)\to\mathbb{R}$ be a C^1-function with $f(0)<0$. To study the problem (0.2) we consider the existence of non-zero solutions of the boundary value problem,

$$(1.1)\qquad -\Delta u(x) \in \hat{f}(u(x)) \ \ a.e.\ x\in\Omega, \qquad u=0 \text{ on } \partial\Omega,$$

where $\hat{f}$ is the multivalued function defined by:

$$\hat{f}(t)=\begin{cases} 0, & \text{if } t<0;\\ [f(0),0], & \text{if } t=0;\\ f(t), & \text{if } t>0.\end{cases}$$

By a solution of (1.1) we mean a function $u\in C^1(\overline{\Omega})\cap C^2(\Omega^*)$ with $\Omega^*=\{x\in\Omega/u(x)\neq \neq 0\}$ and verifying (1.1). (Observe that $\Delta u(x)$ is well-defined in $\Omega^*\cup(\Omega-\overline{\Omega}^*)$).

Notice that all solutions u of (1.1) are non-negative by the maximum principle; so they are solutions of (0.2). However, in contrast with [2] (where the case $f(0)\geqslant 0$ is studied) we cannot deduce that $u>0$ in Ω.

In this section we will assume:

(f_1) $f(0)<0$ and there exists $\theta>0$ such that $f(\theta)=0$, with f increasing in $[0,\theta]$.

(f_2) $f(s)\leqslant \alpha s+\beta$, with $\beta\in\mathbb{R}$, $0\leqslant\alpha<\lambda_1$, where λ_1 denotes the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary conditions.

We will take a positive eigenfunction ϕ_1 associated to λ_1 such that: $\|\phi_1\|_{L^2(\Omega)}=1$. Let $\sigma=\|\phi_1\|_\infty \|\Omega\|^{1/2}$. (We denote $|\Omega|\equiv \operatorname{meas}\Omega$).

(f_3) There exists s_0, s_1, $\gamma\in\mathbb{R}$ such that

i) $\theta<s_0<s_1/\sigma, \gamma>[\lambda_1 s_1^2-2f(0)\,\theta\sigma^2](s_1^2-s_0^2\sigma^2)^{-1}\equiv\gamma^*$;

ii) $f(s)\geqslant 0\ \ \forall s\in(\theta,s_0)$ and $f(s)\geqslant\gamma s\ \ \forall s\in(s_0,s_1)$.

Remarks 1.1. *a*) A sufficient condition to assumption *i*) of (f_3) is

i′) $\theta<s_0<(s_1^2-1)^{1/2}\sigma^{-1}, \gamma>\lambda_1 s_1^2-2f(0)\,\theta\sigma^2$.

b) Notice that $\gamma^*>\lambda_1$, hence the meaning of (f_3) is, roughly, that $f(s)\gg\lambda_1 s$ on a suitable interval (s_0,s_1).

To study (1.1) we consider the sequence of problems:

$$(1.2)\qquad -\Delta u=f_n(u) \text{ in } \Omega, \qquad u=0 \text{ on } \partial\Omega$$

where $f_n\colon\mathbf{R}\to\mathbf{R}$ is of class C^1 and verifies $f_n(t)=f(t)$ $\forall t>1/n$, $f_n(t)=0$ $\forall t\leqslant 0$, $f_n(t)\geqslant f(t)$ $\forall t\in(0,1/n]$, for all $n\in\mathbf{N}$.

Changing β, if it is necessary, we can suppose the next uniform estimate for all f_n:

$$f_n(s) \leqslant \alpha s + \beta: \qquad \forall s \geqslant 0 \text{ and } 0 < \beta. \tag{1.3}$$

PROPOSITION 1.2. *Let us assume* (f_{1-3}). *For all* $n \in \mathbf{N}$ *the problem* (1.2) *has at least two nontrivial classical solutions* $u_n \neq v_n$ *verifying*:

i) $u_n(x), v_n(x) > 0$, $\forall x \in \Omega$. ii) $\|u_n\|_\infty, \|v_n\|_\infty > \theta$.

PROOF. Let $E := H_0^1(\Omega)$ be the usual Sobolev space, $\left(\text{with } \|u\|_E^2 = \int_\Omega |\nabla u(x)|^2\, dx\right)$. We define the C^1-functionals $I_n : E \to \mathbb{R}$ by setting:

$$I_n(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega F_n(u)\, dx, \qquad \forall u \in E,$$

where $F_n(t) = \int_0^t f_n(s)\, ds$.

It is well-known that the critical points of I_n are classical solutions of (1.2) and that (1.3) implies that I_n is coercive and verifies the Palais-Smale condition [3]. Because of this, I_n attains its infimum on a function u_n. Moreover, since $f_n'(0) = 0$, I_n has a local minimum at 0.

On the other hand, let $\phi = s_1 \phi_1 (\|\phi_1\|_\infty)^{-1}$.

By (f_3),

1) $\|\phi\|_{L^2(\Omega)} > \theta |\Omega|^{1/2}$;

2) $\theta < s_0 < \|\phi\|_{L^2(\Omega)} |\Omega|^{-1/2}$ and

$$\gamma > \left[\lambda_1 \int_\Omega \phi^2(x)\, dx - 2f(0)\,\theta|\Omega|\right]\left[\int_\Omega \phi^2(x)\, dx - s_0^2|\Omega|\right]^{-1};$$

3) $f(s) \geqslant 0$ $\forall s \in (\theta, \|\phi\|_\infty)$ and $f(s) \geqslant \gamma s$ $\forall s \in (s_0, \|\phi\|_\infty)$.

By 3),

$$F_n(s) \geqslant \int_0^\theta f(t)\, dt + \frac{\gamma}{2}(s^2 - s_0^2), \qquad \forall s \in (s_0, \|\phi\|_\infty). \tag{1.4}$$

Let $\Omega' = \{x \in \Omega / \phi(x) \geqslant s_0\}$. Notice that $\Omega' \neq \emptyset$, otherwise, $\phi(x) < s_0$ $\forall x \in \Omega$ implies $\|\phi\|_{L^2(\Omega)} < s_0 |\Omega|^{1/2}$, in contradiction with 1).

By (1.4), we have:

$$\int_{\Omega'} F_n(\phi(x))\, dx \geqslant \frac{\gamma}{2}\int_{\Omega'} (\phi^2(x) - s_0^2)\, dx + \left(\int_0^\theta f(t)\, dt\right)|\Omega'|.$$

Moreover,

$$\int_{\Omega - \Omega'} F_n(\phi(x))\, dx \geqslant \int_{\Omega - \Omega'} \left(\int_0^\theta f(t)\, dt\right) dx \geqslant \left(\int_0^\theta f(t)\, dt\right)|\Omega - \Omega'|.$$

Hence,

$$I_n(\phi) = \frac{\lambda_1}{2} \int_\Omega \phi^2(x)\, dx - \int_\Omega F_n(\phi(x))\, dx \leqslant$$

$$\leqslant \frac{\lambda_1}{2} \int_\Omega \phi^2(x)\, dx - \frac{\gamma}{2} \int_{\Omega'} (\phi^2(x) - s_0^2)\, dx - \left(\int_0^\theta f(t)\, dt \right) |\Omega| \leqslant$$

$$\leqslant \frac{\lambda_1}{2} \int_\Omega \phi^2(x)\, dx - \frac{\gamma}{2} \int_\Omega (\phi^2(x) - s_0^2)\, dx - \left(\int_0^\theta f(t)\, dt \right) |\Omega| \equiv \delta_0 .$$

By 2), it follows that $I_n(\phi) \leqslant \delta_0 < 0$. So, all hypotheses of Mountain Pass Theorem [3], are verified and I_n has another critical point v_n. In addition, there results

$$I_n(u_n) \leqslant \delta_0 < 0 < I_n(v_n) \tag{1.5}$$

which implies that $u_n \neq v_n$, are non zero solutions of (1.2). Finally, simple applications of minimum and maximum principles imply i) and ii). ■

In order to obtain solutions of (1.1), we need the next lemma:

LEMMA 1.3. *Under the hypotheses* (f_{1-3}), *the sequences* $\{u_n\}$, $\{v_n\}$ *have subsequences* $\{u_{n_k}\}$, $\{v_{n_k}\}$ *such that* $\{u_{n_k}\} \to u_0$, $\{v_{n_k}\} \to v_0$ *in* $C^{1+\nu}(\overline{\Omega})$, *with* $0 < \nu < 1$.

PROOF. By (1.3), we obtain an a priori estimate $\|u_n\|_E, \|v_n\|_E \leqslant \beta |\Omega|^{1/2} [(1 - \alpha/\lambda_1)\lambda_1]^{-1}$; $\forall n \in \mathbf{N}$, and using usual bootstrap arguments we obtain converging subsequences of $\{u_n\}$, $\{v_n\}$ in $C^{1+\nu}(\overline{\Omega})$ ■.

From now on, we denote $\{u_{n_k}\} \equiv \{u_n\}$ and $\{v_{n_k}\} \equiv \{v_n\}$.

THEOREM 1.4. *Let us assume* (f_{1-3}). *There exist at least two distinct, non-negative and non-zero solutions of* (1.1).

PROOF. Let u_0, v_0 be given by Lemma 1.3. Clearly, u_0, v_0 are non-negative and non-zero by Proposition 1.2-ii) and Lemma 1.3.

Notice that $\lim_{n\to\infty} F_n(t) = F(t)$ $\forall t \in \mathbb{R}$, where

$$F(t) = \begin{cases} 0, & \text{if } t < 0; \\ \displaystyle\int_0^t f(s)\, ds, & \text{if } t \geqslant 0. \end{cases}$$

So, by Lemma 1.3 and the Lebesgue's dominated convergence theorem:

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n(x)|^2\, dx = \int_\Omega |\nabla u_0(x)|^2\, dx$$

and

$$\lim_{n\to\infty} \int_\Omega F_n(u_n(x))\, dx = \int_\Omega F(u_0(x))\, dx .$$

Then $\lim_{n\to\infty} I_n(u_n) = I(u_0)$, where

$$I(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2\, dx - \int_\Omega F(u(x))\, dx .$$

Similar arguments prove $\lim_{n\to\infty} I_n(v_n) = I(v_0)$. Hence, by (1.5) $I(u_0) \leq \delta_0 < 0 \leq I(v_0)$ and $u_0 \neq v_0$.

In order to prove that u_0, v_0 are solutions of (1.1), we observe that

$$\lim_{n\to\infty} f_n(x_n) = f(x) \text{ if } \lim_{n\to\infty} x_n = x > 0.$$

Then, if $\Omega^* = \{x \in \Omega / u_0(x) \neq 0\}$,

$$\lim_{n\to\infty} f_n(u_n(x)) = f(u_0(x)) \ \ \forall x \in \Omega^*.$$

So that, for all $u \in C_0^\infty(\Omega^*)$, the equalities

$$\int_{\Omega^*} \nabla u_n(x) \nabla u(x)\, dx + \int_{\Omega^*} f_n(u_n(x))\, u(x)\, dx = 0$$

and Lemma 1.3 imply

$$\int_{\Omega^*} \nabla u_0(x) \nabla u(x)\, dx + \int_{\Omega^*} f(u_0(x))\, u(x)\, dx = 0\,.$$

In particular, $u_0 \in C^2(\Omega^*)$ and $-\Delta u_0(x) = f(u(x))$, in Ω^*.

Finally, by a Morrey-Stampacchia theorem (see [9, Theorem 3.2.2, p. 69]), we have also $-\Delta u_0(x) = 0$ *a.e.* $\Omega - \overline{\Omega^*}$, and so u_0 is a solution of (1.1).

The same ideas show v_0 is another solution of (1.1). ■

Remark 1.5. Observe that our technique can be combined with some symmetry properties of the domain. More precisely, if Ω is symmetric in the sense of Steiner [8] (*i.e.* Ω is simmetric with respect to a plane, for instance $x_1 = 0$, and convex in the variable x_1), we deduce [7] that u_n, v_n are symmetric (in the sense of Steiner). Hence, their limits u_0, v_0 (which are solutions of (1.1) as it has been proved) are symmetric also.

2. The superlinear case

Our method for study (1.1) can be useful to prove existence of solutions for other hypotheses on f. For instance, the superlinear case. We assume:

(f_4) There exist a_1, $a_2 \geq 0$ such that

$$|f(s)| \leq \begin{cases} a_1 + a_2 |s|^\mu, & \text{if } N > 2, \\ a_1 \exp(\phi(s)), \quad \text{with} \quad \phi(s)\, s^{-2} \to 0 (|s| \to \infty), & \text{if } N = 2, \end{cases}$$

where $0 \leq \mu < (N+2)(N-2)^{-1}$.

(f_5) There exist $\rho > 2$ and $r \geq 0$ such that $0 < \rho F(s) \leq s f(s)$ $\forall s \geq r$.

Theorem 2.1. *Let us assume* (f_1), (f_{4-5}). *Then, the problem* (1.1) *has at least one nonnegative and nonzero solution.*

Proof. Let f_n, F_n, F, I_n and I be functions like in section 1. Since by (f_4) (see [3])

$$\lim_{t\to+\infty} I(t\phi_1) = -\infty,$$

we deduce that there exists $t_0 > 0$ such that $I_n(t_0 \phi_1) \leq I(t_0 \phi_1) < 0$ for all $n \in \mathbf{N}$.

Moreover, f_n satisfies (f_5) and $f_n'(0) = 0$. Then I_n verifies all hypotheses of the

Mountain Pass Theorem [3] with $\bar{e}=t_0\phi_1$ (independently of $n\in\mathbf{N}$). Consequently, it has a critical point u_n such that

$$0<I_n(u_n)\leq \max_{t\in[0,1]} I_n(t\bar{e})\leq \max_{t\in[0,1]} I(t\bar{e})\equiv M\,,\qquad u_n>0 \text{ in } \Omega\,,\qquad \|u_n\|_\infty>\theta\,,$$

for all $n\in\mathbf{N}$.

In order to prove an a priori estimate of u_n, observe that

$$M\geq I_n(u_n)=I_n(u_n)-\frac{1}{\rho}I_n'(u_n)\,u_n=\left(\frac{1}{2}-\frac{1}{\rho}\right)\|u_n\|_E^2+\int_\Omega\left[\frac{f_n(u_n)\,u_n}{\rho}-F_n(u_n)\right]dx\,.$$

Since $f_n\geq f$ and $|F_n(u)-F(u)|\leq -f(0)\,u\ \forall u\geq 0$, we deduce

$$M\geq\left(\frac{1}{2}-\frac{1}{\rho}\right)\|u_n\|_E^2+\int_\Omega\left[\frac{f(u_n)\,u_n}{\rho}-F(u_n)\right]dx+f(0)\int_\Omega u_n\,dx\,.$$

By (f_5), it follows $M\geq(1/2-1/\rho)\|u_n\|_E^2-k\|u_n\|_E-c$ where $k,c>0$, and hence $\{u_n\}$ is bounded in E.

Then a limiting procedure and similar arguments to those of Theorem 1.4 conclude the proof. ■

REMARKS 2.2. *a*) Notice that from the results of [5] if, in addition to (f_1), $(f_{4\text{-}5})$, we assume that f is non-increasing, we can consider a «small» ball $B\subset\Omega$ and a positive solution u of $-\Delta u=f(u)$ in B, $u=0$ on $\partial\Omega$.

Hence, the extension zero in $\Omega-B$ yields a weak solution u of $-\Delta u\in\hat{f}(u)$ *a.e.* in Ω, $u=0$ on $\partial\Omega$.

We remark that this procedure gives us solutions which may be not of class C^1 (see [5, Theorem 1.1]).

b) Observe also that in the case that $\Omega=B(r)$ is the ball of radius r centered at $x=0$ and under the conditions of *a*), we know by [4] that there exists $r^*>0$ such that the problem $-\Delta u=f(u)$ in $B(r)$, $u=0$ on $\partial B(r)$ has no radial nonnegative solutions for $r>r^*$.

On the other side, we have obtained, see Theorem 2.1 and Remark 1.5, the existence of one C^1-radial non-negative solution of the multivalued problem (1.1), for all $r>0$. This proves the existence of an r_0 and a positive radial solution u of $-\Delta u=f(u)$ in $B(r_0)$, $u=0$ on $\partial B(r_0)$ such that $u'(r_0)=0$ (we denote $u(s)=$ $=u(|x|)$, $x\in B(r)$). This kind of solutions differ from those of [5] where $u'(r_0)<0$.

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