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Wiener criterion for degenerate elliptic obstacle problem

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Analisi matematica. — *Wiener criterion for degenerate elliptic obstacle problem.* Nota (*) di Marco Biroli (**) e Umberto Mosco (***), presentata dal Socio L. Amerio.

Abstract. — We give a Wiener criterion for the continuity of an obstacle problem relative to an elliptic degenerate problem with a weight in the A_2 class.

Key words: Variational inequalities; Potential theory; Regularity of weak solutions.

Riassunto. — *Criterio di Wiener per problemi d'ostacolo relativi ad operatori ellittici degeneri.* Si fornisce un criterio tipo Wiener per la continuità della soluzione di un problema di ostacolo relativo ad un operatore ellittico degenere con peso di classe A_2.

1. Introduction

In a recent paper Fabes, Kenig and Serapioni, [4], have worked out a regularity theory for solutions of Dirichlet problems for degenerate 2nd order elliptic operators, generalizing the usual classical theory for uniformly elliptic operators with measurable bounded discontinuous coefficients.

The class of degenerate operators used in this paper is described by a weight function w, belonging to the Muckenhoupt's class A_2 (for the definition of class A_2 see [4]).

The characterization of regular boundary points for this class of degenerate operators has been studied by Fabes, Jerison and Kenig in a second paper, [3], where a generalization of the classical Wiener criterion of Littman-Stampacchia-Weinberger, [6], (concerning uniformly elliptic operators with measurable bounded coefficients) is proved.

In this paper an intrinsic notion of capacity related to the weight w is studied; roughly speaking they deal with the capacity corresponding to the space $H_0^1(\Omega; w)$, which in this theory replaces the usual space $H_0^1(\Omega)$.

Estimates on the modulus of continuity and of energy decay in a regular point have been by Biroli and Marchi in [1], with particular assumptions on the weight w, and in [2] in the general case.

The essential goal this paper is to give pointwise regularity properties (related to the shape of the obstacle) for the solution of an *obstacle* problem for the class of degenerate elliptic operators considered in [7].

For uniformly elliptic operators with bounded measurable coefficients this study has been recently carried out by Frehse and Mosco, [5], Mosco, [7], and is founded on a description of the pointwise behaviour of the obstacle, which is given in the Newtonian capacity sense. In particular they have obtained a Wiener criterion to

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characterize the regular points. Moreover they proved also estimates on the modulus of continuity in a regular point and of the pointwise «energy decay» type.

Here we give an extension of those results to the above mentioned class of degenerate operators, using the theory of w-capacity given in [3].

2. Results

Let Ω be a bounded open set in $\boldsymbol{R}^N$; the space $H^1(\Omega; w)$ is the space of the functions in $L^2(\Omega; w)$ with gradient in $L^2(\Omega; w)$ endowed with the norm $\left(\int_\Omega |u|^2 w\, dx + \int_\Omega |Du|^2 w\, dx\right)^{1/2}$; the space $H_0^1(\Omega; w)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega; w)$.

We recall now the notion of w-capacity:

Definition 1. *Let B be an open set of $\boldsymbol{R}^N$ and K be a compact of B. The w-capacity of K with respect to B (denoted w-cap(K, B)) is defined as the infimum of the integral*

$$\int_B |D_x v|^2 w\, dx$$

taken on all funtions v in $C_0^1(B)$ such that $v \geq 1$ on K.

If A is an openset in B, then the w-capacity of A with respect to B (w-cap(A, B)) is defined as the supermum of w-cap(K, B) for all compact set $K \subset A$.

If E is an arbitrary set in B the external w-capacity of E with respect to B, still denoted as w-cap(E, B), is defined as the minimum of w-cap(A, B) for all open sets $A \subset B$ such that $E \subset A$.

For the properties of the w-capacity we refer to [3]; we remark that a point may have positive capacity (for ex. if weight is given by $|x|^{-\alpha}$, $N-2 < \alpha < N$, the origin is point of positive capacity). A technical implication of this fact is that the usual Kellog argument, by which one proves that the Wiener criterion can be equivalently stated with respect to balls or to anuli, can not be carried out to our weighted capacity.

The definition w-quasi continuity (w-q.c.) is analogous to the usual one in the case of the Newtonian capacity and the w-q.c. of the functions in $H^1(\Omega, w)$ has been proved in [3].

In the following we denote by $w\text{-}\sup_E g$ $\left(w\text{-}\inf_E g,\ w\text{-}\operatorname{osc}_E g\right)$ the essential supremum (infimum, oscillation) of g on the set E.

We say that g is continuous at the point x_0 if $w\text{-}\operatorname{osc}_{B(\rho, x_0)} g \to 0$ as $\rho \to 0$. We observe that the w-oscillation and the oscillation in the a.e. sense are in general different but coincide in the case of w-q.c. functions.

Let now $a_{ij}(x)$, $i,j = 1, 2, \ldots, N$, be measurable functions such that

$$\lambda w |\xi|^2 \leq \sum_{i=1}^N a_{ij}(x)\, \xi_i \xi_j \leq \Lambda w |\xi|^2 \qquad \forall \xi \in \boldsymbol{R}^N, \tag{2.1}$$

where w is a weight of the A_2 Muckenhoupt's class.

We denote

$$a_\Omega(u,v) = \sum_{i=1}^{N} \int_\Omega a_{ij}(x)\, D_j u(x)\, D_i v(x)\, dx \qquad \forall u, v \in H^1(\Omega; w). \tag{2.2}$$

Consider a function $\psi: \boldsymbol{R}^N \to [-\infty, +\infty)$ defined up to set of w-capacity 0 (w-q.e.); a local solution in Ω of the obstacle problem relative to ψ is a function u such that

$$\begin{cases} a_\Omega(u, u-v) \leq 0 \\ u \in H^1(\Omega; w),\ u \geq \psi \qquad w\text{-q.e. in } \Omega \\ \forall v \in H^1(\Omega; w),\ v \geq \psi \qquad w\text{-q.e. in } \Omega,\ u - v \in H_0^1(\Omega; w). \end{cases} \tag{2.3}$$

Definition 2. *We say that x_0 is a regular point of ψ if there exists at least one local solution of the obstacle problem relative to ψ in a neighbourhood of x_0 and every local solution in a neighbourhood Ω of x_0 is continuous at x_0.*

We define now the notion of Wiener point for ψ.

Following [4], [6], given an arbitrary $\varepsilon > 0$ and an arbitrary $\rho > 0$, we consider the one-sided level sets of ψ

$$E(\varepsilon, \rho) = E(\psi, x_0; \varepsilon, \rho) = \{x \in B(\rho, x_0) \mid \psi(x) \geq w\text{-}\sup_{B(\rho, m x_0)} \psi - \varepsilon\},$$

where $m \in (0, 1]$, and their relative capacity

$$\delta(\rho) = \delta(\varepsilon, \rho) = w\text{-cap}\,(E(\varepsilon, \rho), B(2\rho, x_0))[w\text{-cap}\,(B(\rho, x_0), B(2\rho, x_0))]^{-1}.$$

Definition 3. *We say that x_0 is a Wiener point of ψ if (a) or (b) below hold*

(*a*) $\overline{\psi}(x_0) < +\infty$,

(*b*) $\int_0^1 \delta(\varepsilon, \rho)\, d\rho/\rho = +\infty$.

The first result we shall give is the following extension of the Wiener criterion for obstacle problems (given for the case of usual uniformly elliptic operators in [6]) to our class of operators:

Theorem 1. *The point x_0 is a regular point for ψ if and only if it is a Wiener point for ψ.*

We define now the *Wiener modulus of ψ at x_0* by the relation

$$\omega_\sigma(r, R) = \inf\left\{\omega > 0 \mid \omega \exp\left(\int_r^R \delta(\sigma\omega, \rho)\, d\rho/\rho\right) \geq 1\right\}, \qquad \sigma > 0.$$

We observe that x_0 *is a Wiener point if and only if* $\omega_\sigma(r, R) \to 0$ *as* $r \to 0$.

Consider the following function

$$V(r) = w\text{-}\operatorname*{osc}_{B(r, x_0)} u + \left(\int_{B(r, x_0)} |Du|^2\, G_{2q^{-1}r, x_0}\, dx\right)^{1/2}$$

where $G_{2q^{-1}r, x_0}$ denotes the Green function relative to our operator and to the ball $B(2q^{-1}r, x_0) \subset \Omega$, $q \in (0.5^{-1}m)$.

The second result we will give is an extimate on the decay of $V(r)$ as $r \to 0$.

THEOREM 2. *Let u be a local solution in Ω of* (2.3); *then there exists constants c, k and $\beta > 0$ such that*

$$V(r) \le cV(R)\, \omega_\sigma(r,R)^\beta + k\,\sigma\,\omega_\sigma(r,R) \tag{2.4}$$

or every $0 < r < mR$, $B(2q^{-1}R, x_0) \subset \Omega$ and every $\sigma > 0$. The constants c, k depends only $N, w, \lambda/\Lambda$, the constant β only on $N, w, \lambda/\Lambda$ and possibly on m.

The inequality (2.4) gives us an estimate on the modulus of continuity of u in a Wiener point.

Let us point out a special case.

Denote

$$F = F_\psi = \{x \mid \psi(x) > -\infty\}$$

and

$$\begin{aligned} B^F(\rho, x_0) &= B(\rho, x_0) \cap F \qquad &&\text{if } w\text{-cap}\,(B(\rho, x_0) \cap F) > 0\\ B^F(\rho, x_0) &= B(\rho, x_0) &&\text{if } w\text{-cap}\,(B(\rho, x_0) \cap F) = 0\,. \end{aligned}$$

Define

$$W_F(r,R) = \exp\left(-\int_r^R w\text{-cap}\,(B^F(\rho, x_0), B(2\rho, x_0))\right)\Big/(w\text{-cap}\,(B(\rho, x_0), B(2\rho, x_0)))\,dr\,.$$

Then

COROLLARY 1. *Under the same assumptions of Theorem 2 and with the same constants as in that theorem we have*

$$V(r) \le c\,V(R)\; W_F(r,R)^\beta + k\,w\text{-}\operatorname*{osc}_{B(R,x_0)\cap F}\psi\,, \tag{2.5}$$

where $0 < r < mR$, $B(4q^{-1}R, x_0) \subset \Omega$.

We observe that in (2.5) the effects of the set F and of the obstacle ψ. In the following we denote by $w\text{-}\sup_E g \left(w\text{-}\inf_E g, w\text{-}\operatorname*{osc}_E g\right)$ the appear separately.

An interesting particular case of Corollary 1 is when $x_0 \in F$ and w-cap $\{x_0\} > 0$.

COROLLARY 2. *Let w-cap $\{x_0\} > 0$ and $x_0 \in F$. If ψ is continuous in F at x_0; then x_0 is a regular point for ψ. Moreover the following estimate holds*

$$V(r) \le c\,V(R)\left\{\int_0^r s^2/w(B(s,x_0))(ds/s)\left[\int_0^R s^2/w(B(s,x_0))(ds/s)\right]^{-1}\right\} + k\,w\text{-}\operatorname*{osc}_{B(R,x_0)}\psi\,.$$

REFERENCES

[1] BIROLI M. and MARCHI S., 1986. *Wiener estimates at boundary points for degenerate elliptic equations.* Boll. U.M.I., 6, 5(B): 689-706; Correction, 1988, Boll. U.M.I., 2-B, 7: 713.

[2] BIROLI M. and MARCHI S., 1989. *Wiener estimates for degenerate elliptic equations II.* Diff. Int. Eq., 2, 4: 511-523.

[3] FABES E., JERISON D. S. and KENIG C., 1982. *The Wiener test for degenerate elliptic equations.* Ann. Inst. Fourier, 3: 151-183.

[4] FABES E., KENIG C. and SERAPIONI R., 1982. *The local regularity of solutions of degenerate elliptic equations.* Comm. in PDE, 7, 1: 77-116.

[5] FREHSE J. and MOSCO U., 1985. *Wiener obstacles.* In «Seminar on nonlinear partial differential equation», College de France, ed. by BREZIS H. and LIONS J. L., VI, Pitman.
[6] LITTMAN W., STAMPACCHIA G. and WEINBERGER H., 1963. *Regular points for elliptc equations with discontinuous coefficients.* Ann. Sc. Norm Sup. Pisa, 17: 47-77.
[7] MOSCO U., 1987. *Wiener criterion and potential estimates for the obstacle problem.* Indiana Un. Math. J., 36: 455-494.