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On holomorphic isometries for the Kobayashi and Carathéodory distances on complex manifolds

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Geometria. — *On holomorphic isometries for the Kobayashi and Carathéodory distances on complex manifolds.* Nota di SERGIO VENTURINI, presentata (*) dal Corrisp. E. VESENTINI.

ABSTRACT. — It is shown that under certain conditions every holomorphic isometry for the Carathéodory or the Kobayashi distances is an isometry for the corrisponding metrics. These results are used to give a characterization of biholomorphic mappings between convex domains and complete circular domains.

KEY WORDS: Complex manifolds, Convex and complete circular domains, Carathéodory and Kobayashi distances and metrics.

RIASSUNTO. — *Isometrie olomorfe per le distanze di Kobayashi e Carathéodory sulle varietà complesse.* Si dimostra che, sotto opportune condizioni, ogni isometria olomorfa per le distanze di Carathéodory o di Kobayashi è una isometria per le rispettive metriche. Si applicano questi risultati allo studio dei biolomorfismi tra domini convessi e domini circolari completi.

1. INTRODUCTION.

For every connected complex manifold M let k_M and c_M be respectively the Kobayashi and Carathéodory (pseudo)distances on M and let $\varkappa_M$ and γ_M be the corrisponding infinitesimal (pseudo) metrics. For the definition of these objects and their principal properties see *e.g.* [6].

Given M and N connected complex manifolds we call a holomorphic mapping $F: M \to N$ a K-isometry at $p \in M$ if

$$k_N(F(q)) = k_M(q, p)$$

for every $q \in M$ and a K-infinitesimal isometry if

$$\varkappa_N(F(p), dF(p)(v)) = \varkappa_M(p, v)$$

for every $v \in T_M M$.

We define holomorphic C-isometries and C-infinitesimal isometries as holomorphic mappings satisfying the previous equalities with the Kobayashi distances and metrics replaced by the Carathéodory ones.

In this note we prove that every holomorphic C-isometry is a C-infinitesimal isometry (theorem 2) and, under some additional hypotheses on M, that every holomorphic K-isometry is a K-infinitesimal isometry (theorem 3).

The above results are used to give a characterization of biholomorphic mappings between convex and circular domains of $\mathbf{C}^n$ as isometries or infinitesimal isometries at one point, improving some results by Patrizio [9].

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2. Complex geodesics.

Let Δ be the unit disk of $\boldsymbol{C}$. For every $z \in \Delta$ and $v \in \boldsymbol{C} \cong T_C \Delta_z$ let

$$\langle v \rangle_z = |v|/(1 - |z|^2)$$

be the length of the tangent vector v to z computed in terms of the Poincaré metric and let

$$\omega : \Delta \times \Delta \rightarrow \boldsymbol{R}^+$$

be the associated distance.

Then we have $\omega = k_\Delta = c_\Delta$ and $\langle \cdot \rangle = \varkappa_\Delta = \gamma_\Delta$ (see [6]).

In [14] Vesentini proved the following result: let M be a complex manifold and let $f: \Delta \rightarrow M$ be a holomorphic mapping. If there exist two distinct points z^0 and w^0 in Δ such that

$$c_M(f(z^0), f(w^0)) = \omega(z^0, w^0),$$

or a point $z^0 \in \Delta$ and $v^0 \in \boldsymbol{C}$, $v^0 \neq 0$, such that

$$\gamma_M(f(z^0), df(z^0)(v^0)) = \langle v^0 \rangle_{z^0}$$

then the first equality holds for every choice of z and w in Δ and the second one for every choice z in Δ and v in $\boldsymbol{C}$.

Vesentini calls such mappings complex geodesics.

Since we work with manifolds for which the Kobayashi and the Carathéodory distances and metrics do not necessarily coincide we call these mappings C-complex geodesics and call K-(infinitesimal) complex geodesics the holomorphic mappings which are K-(infinitesimal) isometries at the point $0 \in \Delta$ (by the result of Vesentini is unnecessary to distinguish between C-complex geodesics and C-complex infinitesimal geodesics).

As pointed out by Vigué [17] there are K-complex infinitesimal geodesics which are not K-complex geodesics.

Now we prove that the converse holds, *i.e.* that every K-complex geodesic is a K-infinitesimal complex geodesic.

We need some preliminaries.

Lemma 1. *Let M be a connected complex manifold. Let $I = [0, 1]$ be the unit interval and let $t_0 \in I$.*

If θ, $\gamma : I \rightarrow M$ are C^1 arcs such that $\theta(t_0) = \gamma(t_0)$ and $\theta'(t_0) = \gamma'(t_0)$ then

$$\lim_{s \rightarrow t_0} k_M(\theta(s), \gamma(s))/|s - t_0| = 0.$$

Proof. If the manifold M is a domain in a Banach space then the proof is in [4]. For the general case let θ and γ be as in the hypoteses. Let U be a n open neighbourhood of $p = \theta(t_0) = \gamma(t_0)$ in M biholomorphic to a domain in a Banach space. Then we have

$$\lim_{s \rightarrow t_0} k_M(\theta(s), \gamma(s))/|s - t_0| \leqslant \lim_{s \rightarrow t_0} k_U(\theta(s), \gamma(s))/|s - t_0| = 0.$$

The following proposition generalizes a result in [4].

PROPOSITION 1. *Let M be a complex manifold. Let* $\theta:[0,1]\to M$ *be a* C^1 *arc. Then, for every* $t\in[0,1]$ *we have*

$$\limsup_{s\to t} k_M(\theta(s),\theta(t))/|s-t| \leq \varkappa_M(\theta(t),\theta'(t)).$$

PROOF. Let $t\in[0,1]$; put $p=\theta(t)$ and $u=\theta'(t)$. Let $\varepsilon>0$. There exists a holomorphic map $f:\Delta\to M$ and $v\in\mathbf{C}$ such that $f(0)=p$, $f'(0)=u$ and $\langle v\rangle_0=|v|<\varkappa_M(u)+\varepsilon$. Let $\sigma:\mathbf{R}\to\Delta$ be the affine geodesic for the Poincaré metric such that $\sigma(t)=0$ and $\sigma'(t)=v$ and let $\gamma=f\circ\sigma$. Then we have $\theta(t)=\gamma(t)$ and $\theta'(t)=\gamma'(t)$. Thus

$$k_M(\theta(s),\theta(t))\leq k_M(\theta(s),\gamma(s))+k_M(\gamma(s),\gamma(t))\leq k_M(\theta(s),\gamma(s))+\omega(\sigma(s),\sigma(t))=$$
$$=k_M(\theta(s),\gamma(s))+|t-s|\,|v|\leq k_M(\theta(s),\gamma(s))+|t-s|(\varkappa_M(p,u)+\varepsilon).$$

By lemma 1 we have

$$\lim_{s\to t} k_M(\theta(s),\gamma(s))/|s-t|=0,$$

hence

$$\limsup_{s\to t} k_M(\theta(s),\theta(t))/|s-t|\leq \varkappa_M(p;u)+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary the thesis follows.

THEOREM 1. *Let M be a connected complex manifold and let* $f:\Delta\to M$ *be a holomorphic mapping. Suppose that there exist two distinct points* z^0 *and* w^0 *in* Δ *such that*

$$k_M(f(z^0),f(w^0))=\omega(z^0,w^0). \tag{1}$$

Let S be the arc of the Riemannian geodesic for the Poincaré metric joining z^0 *with* w^0. *Then, for every choice of z and w in S we have*

$$k_M(f(z),f(w))=\omega(z,w) \tag{2}$$

and for every $z\in S$ *and* $v\in\mathbf{C}$

$$\varkappa_M(f(z),df(z)(v))=\langle v\rangle_{z^0}. \tag{3}$$

PROOF. Let $d=\omega(z^0,w^0)$ and let $\theta:[0,1]\to\Delta$ be the (unique) affine geodesic parametrized in such a way that $\theta(0)=z^0$, $\theta(1)=w^0$ and whose image is S. Let z and w be two arbitrary points lying in S and let t, $s\in[0,1]$ be such that $\theta(t)=z$ and $\theta(s)=w$, chosen in such a way that $t\leq s$. Then we have

$$k_M(f(z^0),f(z))\leq\omega(z^0,z),$$
$$k_M(f(z),f(w))\leq\omega(z,w),$$
$$k_M(f(w),f(w^0))\leq\omega(w,w^0),$$
$$k_M(f(z^0),f(w^0))\leq k_M(f(z^0),f(z))+k_M(f(z),f(w))+k_M(f(w),f(w^0))\leq$$
$$\leq\omega(z^0,z)+\omega(z,w)+\omega(w,w^0)=\omega(z^0,w^0)=k_M(f(z^0),f(w^0)).$$

and (2) follows.

For every $t\in[0,1]$, by proposing 1 we have

$$\varkappa_M(f(\theta(t)),df(\theta(t))(\theta'(t)))\geq\limsup_{s\to t} k_M(f(\theta(s)),f(\theta(t)))/|s-t|=$$
$$=\limsup_{s\to t}\omega(\theta(s),\theta(t))/|t-s|=\langle\theta'(t)\rangle_{\theta(t)}=\varkappa_\Delta(\theta(s);\theta'(t))\geq\varkappa_M(f(\theta(t)),df(\theta(t))(\theta'(t))),$$

and (3) follows.

The following corollaries are immediate consequences of theorem 1.

COROLLARY 1. *Let $f: \Delta \to M$ be a holomorphic mapping. If there exists a point $z^0 \in \Delta$, $z^0 \neq 0$, such that*

$$k_M(f(z^0), f(0)) = \omega(z^0, 0).$$

then the mapping f is a K-infinitesimal complex geodesic.

COROLLARY 2. *Every K-complex geodesic is a K-infinitesimal complex geodesic.*

COROLLARY 3. *Let $f: \Delta \to M$ be a holomorphic mapping and suppose that there exists r, $0 < r < 1$, such that*

$$k_M(f(z), f(0)) = \omega(z, 0)$$

for every $z \in \Delta$ with $|z| = r$. Then this equality holds for every $z \in \Delta$ with $|z| \leq r$ and at these points the mapping f is a K-infinitesimal isometry.

3. ISOMETRIES AND INFINITESIMAL ISOMETRIES.

In this section the relationships between holomorphic isometries and infinitesimal isometries for the Carathéodory and Kobayashi distances and metrics are investigated.

THEOREM 2. *Let M and N be connected complex manifolds and $p \in M$ a point. Then every holomorphic C-isometry at p is a C-infinitesimal isometry at p.*

Conversely, if for every $q \in M$ there is a complex geodesic $f: \Delta \to M$ such that p and q lie in $f(\Delta)$, then every holomorphic C-infinitesimal isometry at p is a C-isometry at p.

PROOF. The second part of the theorem is due to Vigué [16]. We prove the first part.

Let $F: M \to N$ be a holomorphic C-isometry at p. Let $v \in T_C M_p$. Let $\theta: [0,1] \to M$ be a C^1 curve such that $\theta(0) = p$ and $\theta'(0) = v$. Since the Carathéodory pseudometric is the derivative of the Carathéodory distance [10, 4] we have

$$\gamma_N(F(p), dF(p)(v)) = \gamma_N(F(\theta(0)), dF(\theta(0))(\theta'(0))) =$$
$$= \lim_{t \to 0} c_N(F(\theta(t)), F(p))/t = \lim_{t \to 0} c_M(\theta(t), p)/t = \gamma_M(p, v).$$

Since $v \in T_C M$ is arbitrary the mapping F is a C-infinitesimal isometry.

REMARK. The hypotheses on the manifold M in the second part of the theorem are satisfied when M is a convex bounded domain of $\mathbf{C}^n$ [7, 8, 13].

THEOREM 3. *Let M and N be connected complex manifolds. Let $p \in M$ be a point such that for every $v \in T_C M_p$ there exists a K-geodesic $f: \Delta \to M$ with $f(0) = p$ and $f'(0) = \lambda v$, $\lambda \in \mathbf{R}_+$, and let $F: M \to N$ be a holomorphic mapping. Suppose that there exists r, $r > 0$, such that for every $q \in M$ with $k_M(p, q) = r$ we have $k_N(F(p), F(q)) = r$. Then the mapping F is a K-infinitesimal isometry.*

PROOF. Let $v \in T_C M_p$. Let $f: \Delta \to M$ be a K-complex geodesic as in the hypothesis. By the definition of Kobayashi pseudometric we have

$$\varkappa_M(p, v) \leq \lambda^{-1}.$$

Consider the mapping $g = F \circ f: \Delta \to M$. Let $z^0 \in \Delta$ be a such that $\omega(0, z^0) = r$ and let $q = f(z^0)$. Being f a K-complex geodesic we have

$$k_M(p, q) = k_M(f(0), f(z^0)) = \omega(0, z^0) = r,$$

hence

$$k_N(g(0), g(z^0)) = k_N(F(p), F(q)) = r = \omega(0, z^0).$$

By corollary 1 of theorem 1 we have

$$\varkappa_M(p, v) \geqslant \varkappa_N(F(p), dF(p)(v)) = \varkappa_N(g(0), \lambda^{-1} g'(0)) = \lambda^{-1} \varkappa_N(g(0), g'(0)) = \lambda^{-1} \geqslant \varkappa_M(p, v).$$

Since $v \in T_C M_p$ is arbitrary, the assertion follows.

Remark. The hypotheses on the complex manifold M are satisfied if M is a convex set of a complex Banach space and $p \in M$ is arbitrary (see [7, 8, 13], for the finite dimensional case and [2] for the general case).

4. Biholomorphic mappings and isometries.

In this section we shall deal only with domain in $\mathbf{C}^n$. Let D be such a domain and $p \in D$. Then $K_p(D)$ and $C_p(D)$ will stand respectively for the indicatrices of the Kobayashi and Carathéodory metrics at the point p. If $0 \in D$ we denote $K_0(D)$ and $C_0(D)$ respectively by $K(D)$ and $C(D)$.

Identifying the complex tangent space to D at p with $\mathbf{C}^n$, the domains $K_p(D)$ and $C_p(D)$ are complete circular domains, $C_p(D)$ is convex and $C_p(D)\, K_p(D)$.

The main result of this section is to give a characterization of biholomorphic mappings between some particular domains improving some results by Patrizio given in [9].

The following lemma is due to Vigué [16]:

Lemma. *Let D be a complete circular domain. Then $C(D)$ is the convex hull of D.*

Proof. By [1], $C(D) \supseteq D$. Let D' be the convex hull of D. Since $C(D)$ is a convex domain, $C(D) \supseteq D'$. Since D is complete circular so is D'. Again by [1], $C(D') = D' \supseteq C(D)$.

Theorem 4. *Let D be a convex bounded domain and D' be complete circular and let $p \in D$. Let $F: D \to D'$ be a holomorphic mapping such that $F(p) = 0$. Then the following conditions are equivalent*:

1) *F is a biholomorphic mapping*;
2) *F is a C-infinitesimal isometry at p*;
3) *F is a K-infinitesimal isometry at p*;
4) *F is a C-isometry at p*;
5) *F is a K-isometry at p*;
6) *there exists r, $r > 0$, such that for every $q \in D$, $c_D(p, q) = r$ implies $c_{D'}(F(p), F(q)) = r$*;
7) *there exists r, $r > 0$, such that for every $q \in D$, $k_D(p, q) = r$ implies $k_{D'}(F(p), F(q)) = r$.*

PROOF. The equivalence between 1), 2) and 4) is proved in [16].

It is obvious that 1) implies 5) and 5) implies 7) and also that 4) implies 6).

By theorem 3 it follows that 7) implies 3).

To complete the proof we show that 6) implies 7) and that 3) implies 2).

Suppose that 6) holds. Let $q \in D$ be such that $k_D(p,q) = r$. Since D is convex $c_D = k_D$ (see [7, 8]), hence

$$k_{D'}(F(p), F(q)) \leqslant c_{D'}(F(p), F(q)) = r = k_D(p,q) \leqslant k_{D'}(F(p), F(q)),$$

and 7) holds.

Suppose now that 3) holds, that is $dF(p)(K_p(D)) = K(D')$. In order to prove 2), *i.e.* that is $dF(p)(C_p(D)) = C(D')$, it suffices to show that $K_p(D) = C_p(D)$ and $K(D') = C(D')$. The first equality holds since D is a convex bounded domain (see [7, 8, 13]). For the second one by Barth [1] we have $K(D') \supseteq D'$. The domain $K(D')$, as image of the convex set $K_p(D) = C_D(D)$ under the linear mapping $dF(p)$, is convex, hence, by the lemma, $K(D') \supseteq C(D')$. Since the other inclusion holds in general the assertion follows.

5. Further remarks and examples.

Let D be a complete bounded circular domain and let $m: \boldsymbol{C}^n \to \boldsymbol{R}^+$ be the Minkowsky functional associated to D. Because D is open, m is upper semicontinuous.

The domain D is pseudoconvex if and only if m is plurisubharmonic [1]. In this case [1] for $v \in \boldsymbol{C}^n$ we have

$$\varkappa_D(0, v) = m(v) \tag{5}$$

and for $z \in D$

$$k_D(0, z) \leqslant \omega(0, m(z)). \tag{6}$$

For every $\zeta \in \boldsymbol{C}^n$ with $m(\zeta) = 1$ let $f_\zeta: \Delta \to D$ de defined by $f_\zeta(z) = z\zeta$.

By (5) every such a f_ζ is a K-infinitesimal geodesic (see also [9]). By (6) we have

$$k_D(f_\zeta(0), f(z)) \leqslant \omega(0, m(z\zeta)) = \omega(0, z). \tag{7}$$

If ζ is a point of discontinuity for m, being k_D continuous in D, by homogeneity of m the first inequality in (7) is strict for every $z \in \Delta \setminus \{0\}$ and hence f_ζ is not a K-geodesic.

The following concrete example is due to Barth [1]. Consider

$$D = \{(z, w) \in \boldsymbol{C}^2 | m(z,w) < 1\},$$

where

$$m(z, w) = \exp\left(\max\left(\log|z|, 1 + \sum_{n=1}^{+\infty} 2^{-n} \log|nw - z|\right)\right),$$

and $\zeta = (1, 0)$.

The corollary of 3 of §1 can be generalized as follows:

THEOREM 5. *Let (M, d_M) and $(N; d_N)$ be connected metric spaces and suppose M locally compact, complete and the distance d_M additive* (see [6] or [11]). *Let $p \in M$. Let $F: M \to N$ be a mapping such that*

$$d_N(F(p), F(q)) \leqslant d_M(p, q)$$

for every $q \in M$ and suppose that for a fixed r, $r > 0$,

$$d_N(F(p), F(q)) = d_M(p, q)$$

for every $q \in M$ with $d_M(p,q) = r$. Then the above equality holds for every $q \in M$ with $d_M(p,q) \leq r$.

PROOF. Let $q \in M$ with $d_M(p,q) \leq r$. By the theorem of Hopf-Rinow (see [11]) there exists a isometry $\theta : \boldsymbol{R} \to M$ for the distance d_M such that $\theta(0) = p$ and $q \in S = \theta[0, r]$. By the same argument used in the proof of the first part of theorem 1 we see that the mapping F is an isometry on S and hence, in particular,

$$d_N(F(p), F(q)) = d_M(p, q) .$$

The assertion follows.

The hypotheses of the theorem are clearly satisfied in the case in which M and N are connected complex manifolds endowed with the Kobayashi distance, M is finite dimensional and complete hyperbolic and F is a holomorphic mapping.

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