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Analisi matematica. — *Geodesics on typical convex surfaces.* Nota di PETER MANFRED GRUBER, presentata (*) dal Socio G. CIMMINO.

In memoriam Antonio Pignedoli (1918-1989)

ABSTRACT. — Using Baire categories uniqueness of geodesic segments and existence of closed geodesics on typical convex surfaces are investigated.

KEY WORDS: Convex surfaces; Geodesics; Uniqueness of geodesic segments; Closed geodesics.

RIASSUNTO. — *Linee geodetiche su superfici convesse tipiche.* Si investigano problemi delle unicità di segmenti geodetici e della esistenza di linee geodetiche chiuse su superfici convesse che sono tipiche nel senso delle categorie Baire.

1. INTRODUCTION AND STATEMENT OF RESULTS

The investigation of typical elements of a space in the sense of Baire categories has a long history starting with results on spaces of continuous functions. In convexity the appearance of such results is of a more recent date, see the surveys [6, 14]. This article deals with geodesics on convex surfaces.

Theorems of Zamfirescu [13] show that geodesic segments or shortest paths on typical convex surfaces can behave quite unexpectedly. For example, he proves that most points on a typical convex surface are not relative interior points of any geodesic segment. Our first result indicates a more regular behaviour. For most convex surfaces most pairs of points are connected by a unique geodesic segment. The second result shows that this cannot be extended to all pairs of points, at least in dimension 3. Finally we prove that most convex surfaces in dimension 3 contain no simple closed geodesic. This contrasts a famous theorem of Lusternik and Schnirelman [9] saying that on a sufficiently smooth topological sphere in dimension 3 there are at least 3 distinct simple closed geodesics. See also [8]. (The concepts of geodesics in convexity and in differential geometry coincide for sufficiently smooth convex surfaces.)

A *convex body* in d-dimensional euclidean space $\mathbb{E}^d$ is a compact convex subset of $\mathbb{E}^d$ with non-empty interior. Its boundary is a (*closed*) *convex surface*. Call a convex surface *polytopal* if the underlying convex body is a polytope. By a *geodesic segment* or *shortest path* on a convex surface S we understand a continuous curve on S connecting two points of S and having minimal euclidean length among all such curves. For any pair of points $p, q \in S$ there exists a geodesic segment on S connecting p, q. Let $\rho_S(p, q)$ denote its length. Then ρ_S is a metric on S, called the *intrinsic* or *geodesic metric*. The topology of S induced by ρ_S coincides with the relative topology of S as a subset of $\mathbb{E}^d$. Hence the metric space $\langle S, \rho_S \rangle$ is complete. See [1, 4].

(*) Nella seduta del 22 giugno 1988.

On the space of non-empty compact subsets of $\mathbb{E}^d$ define a metric δ by

$$\delta(C,D) = \max\left\{ \sup_{x \in D} \inf_{y \in D} |x-y|, \sup_{y \in D} \inf_{x \in C} |x-y| \right\}$$

where $|\ |$ denotes the euclidean norm on $\mathbb{E}^d$. δ was first defined and put to use by Hausdorff and Blaschke. The selection theorem of Blaschke yields that the space $\boldsymbol{S}$ of convex surfaces in $\mathbb{E}^d$ endowed with the topology induced by δ is locally compact. See [2, 5].

In a complete metric or locally compact space a *meager set* or a *set of first Baire category*, that is a countable union of nowhere dense sets, can be considered small by Baire's category theorem. When speaking of *most* or of *typical* elements of such a space we mean all elements, except those in a meager set. See [7, 10].

THEOREM 1. *For most convex surfaces S in $\mathbb{E}^d$ most pairs of points $(p,q) \in S \times S$ are connected by a unique geodesic segment.*

Here the cartesian product $S \times S$ is considered as a compact subset of $\mathbb{E}^{2d}$. One must not expect to be able to substantially improve upon Theorem 1, since for $d = 3$ we have the following result:

THEOREM 2. *For most convex surfaces S in $\mathbb{E}^d$ and any $p \in S$ the set of points $q \in S$ which are connected with p by at least two distinct geodesic segments is dense in S.*

A *geodesic* G on a convex surface S is defined as a continuous curve on S which locally is a geodesic segment. It has always a parametrization in terms of arc length. G is *simple* if in a parametrization in terms of arc length points corresponding to different values of the parameter are distinct except, possibly, for the endpoints. A simple geodesic G is *closed* if it has finite lenght, its end points coincide and are in the relative interior of a geodesic segment contained in G.

More general than the concept of geodesics is that of quasigeodesics; these are «limits» of geodesics, see *e.g.* [1], p. 373, [4], p, 114. Using the theorem of Lusternik and Schnirelman [9] referred to above, Pogorelov [11] proved in the convex case the following extension of it: On any convex surface in $\mathbb{E}^3$ there are at least three distinct closed quasigeodesics. Our last result shows that in Pogorelov's theorem one cannot replace quasigeodesics by geodesics.

THEOREM 3. *Most convex surfaces in $\mathbb{E}^3$ contain no simple closed geodesic.*

2. PROOF OF THEOREM 1

We may clearly assume that $d \geq 3$. To facilitate the proof first some needed tools are collected.

For the moment being we call a geodesic as defined in sect. 1 a *geodesic in the sense of convexity*. By a *geodesic in the sense of differential geometry* on a convex surface of class $\mathbf{C}^2$ we understand a curve satisfying the corresponding differential equation; see *e.g.* [3], p. 178. The euclidean length of a continuous curve on a convex surface is defined as the supremum of the euclidean lengths of the inscribed polygons. For a convex surface of class $\mathbf{C}^2$ a formally slightly different definition of Siegel [12] leads to the same value for the euclidean length. (This can be seen by representing the convex surface locally in the form $\xi_d = f(\xi_1, \ldots, \xi_{d-1})$ with $f \in \mathbf{C}^2$.) Thus a result of Siegel [12], p. 86, can be stated as follows:

(1) Let S be a convex surface of class $\mathbf{C}^2$ with positive gaussian curvature. Then any geodesic segment on S in the sense of convexity is a geodesic in the sense of differential geometry.

In the following it will always be clear from the context which type of geodesic we consider.

A sequence of continuous curves in $\mathbb{E}^d$ is said to *converge* to a continuous curve if there are parametrizations of these curves and of the limiting curve, all defined on the same compact interval, for which the convergence is uniform. Note that convergence in this sense implies convergence of (the sets determined by) the curves in the sense of the metric δ. Convergence for a sequence of convex surfaces is defined by means of the metric δ, see the definition of the topology on the space $\mathbf{S}$ of all convex surfaces in sect. 1. The next proposition is taken from Alexandrow [1], p. 106; see also [4], pp. 75,81.

(2) Let (S_i) be a sequence of convex surfaces converging to a convex surface S and for each i let G_i be a geodesic segment in S_i connecting points $p_i, q_i \in S_i$. Then there is a subsequence G_{i_k} converging to a geodesic segment G on S and subsequences (p_{i_k}), (q_{i_k}) converging to points p resp. q in S which are connected by G.

The definition of δ yields the next proposition:

(3) Let (S_i) be a sequence of convex surfaces converging to a convex surface S. If $p \in S$ then there are points, $p_i \in S_i$ such that (p_i) converges to p. Conversely, the limit of a convergent sequence (p_i) where $p_i \in S_i$ is in S.

Since a metric is continuous with respect to the topology induced by it, the following holds:

(4) Let (C_i), (D_i) be sequences of non-empty compact sets in $\mathbb{E}^d$ converging to compact sets C and D, respectively, and let $\varepsilon > 0$. If $\delta(C_i, D_i) \geqslant \varepsilon$ for all i, then $\delta(C, D) \geqslant \varepsilon$ too.

Having available the above tools, the proof of Theorem 1 is comparatively simple.

We first prove the following:

(5) Let S be a convex surface of class C^3 with positive gaussian curvature and let G be a geodesic segment in S connecting points $p, q \in S$. Then if $r, s \in G \setminus \{p, q\}$, the subsegment of G connecting r, s is the unique geodesic segment in S connecting r, s.

Let H be a geodesic segment in S connecting r, s and suppose that $H \not\subset G$. Then we may choose distinct points $t, u \in G \cap H$ such that the subsegment K of H connecting t, u intersects G at t, u only. By exchanging t, u if necessary we may assume that p, t, u, q are in this order on G. We distinguish two cases:

(i) The tangents of G and K at t coincide. Since G and K both satisfy the same differential equation and the same initial conditions at t and since the differential equations is of class C^1, we have that $K \subset G$, a contradiction.

(ii) The tangents of G and K at t are distinct. Then the subsegment of G from p to t, the geodesic segment K and the subsegment of G from u to q form a geodesic segment in S connecting p, q which is not of class C^2, in contradiction to (1). This proves (5). The next proposition is an immediate consequence of (5).

(6) Let S be a convex surface of class C^3 with positive gaussian curvature. Then there is a set of pairs (r, s) dense in $S \times S$ such that r, s are connected by a unique geodesic segment in S.

For $k, m = 1, 2, \ldots$, let

$\boldsymbol{S}_{km} = \{S \in \boldsymbol{S}$: There is a pair $(p, q) \in S \times S$ such that for any pair $(r, s) \in S \times S$ with $(|p - r|)^2 + |q - s|^2)^{1/2} < 1/k$ there are two geodesic segments in S connecting r, s and having distance $\geqslant 1/m\}$.

Here distance means distance in the sense of the metric δ of the sets determined by the geodesics. Using (2), (3) and (4) it is routine to show that

$$\boldsymbol{S}_{km} \text{ is closed in } \boldsymbol{S}.$$

By (6),

$$\boldsymbol{S}_{km} \text{ has empty interior in } \boldsymbol{S}.$$

Thus

$$\bigcup_{k,m} \boldsymbol{S}_{km} \text{ is meager is } \boldsymbol{S}. \tag{7}$$

For $S \in \boldsymbol{S}$ and $m = 1, 2, \ldots$, let

$A_m(S) = \{(p, q) \in S \times S$: There are two geodesic segments in S connecting p, q and having distance $\geqslant 1/m\}$.

Clearly,

(8) $\bigcup_m A_m(S) = \{(p,q) \in S \times S$: There are two distinct geodesic segments in S connecting $p,q\}$.

A simple version of (2) ($S_1 = S_2 = \ldots = S$) together with (4) yields that $A_m(S)$ is closed in $S \times S$. If $A_m(S)$ has non-empty interior in $S \times S$, then the definition of $\mathbf{S}_{km}$ shows that $S \in \mathbf{S}_{km}$ for all sufficiently large k. Thus $S \in \mathbf{S} \setminus \bigcup_{k,m} \mathbf{S}_{km}$ implies that $A_m(S)$ has empty interior for all m. Hence

$$S \in \mathbf{S} \setminus \bigcup_{k,m} \mathbf{S}_{km} \text{ implies that } \bigcup_m A_m(S) \text{ is meager in } S.$$

Together with (7) and (8) this confirms Theorem 1.

3. Proof of theorem 2.

In this section let $d = 3$. We first put together some background material. Zamfirescu [13] proved that

(9) on most convex surfaces most points are *endpoints*,

that is to say, they are not relative interior points of geodesic segments. The next proposition is due to Alexandrow [1], p. 48; see also [4], p. 98.

(10) Let G, H be geodesic segments on a convex surface. Then precisely one of the following holds:
(i) $G \cap H = \emptyset$;
(ii) $G \cap H$ consists of exactly one point;
(iii) G, H have precisely their (ordinary) endpoints in common;
(iv) $G \cap H$ is a subsegment of both G and H, one endpoint of which is an endpoint of G and the other one an endpoint of H.

If a geodesic segment on a convex surface is the unique geodesic segment connecting its endpoints p, q we call it *unique* and denote it $G(p,q)$. Proposition (2) yields the following:

(11) Let $G(p, t_0)$ be a unique geodesic segment on a convex surface S. Then any geodesic segment in S connecting $p, t_i \in S$ is arbitrarily close to $g(p, t_0)$(in the sense of the metric δ) assuming that t_i is sufficiently close to t_0.

From the definition of δ we easily conclude the following:

(12) Let (C_i) be a sequence of non-empty compact sets converging to a compact set C in the sense of the metric δ. If the C_i's are all contained in a fixed closed set D, then C is also contained in D.

By (9) it is sufficient for the proof of Theorem 2 to show the following proposition.

(13) Let S be a convex surface with a dense set of endpoints, let $p, q \in S$ be distinct and let N be a neighbourhood of q in S with $p \notin N$. Then there is a point $t \in N$ which is connected with p by two distinct geodesic segments.

To prove this, assume the contrary, *i.e.*

(14) any point of N is connected with p by a unique geodesic segment.

Let $r, s \in N, r \neq s$ be endpoints which are so close to q that a geodesic segment $G(r, s)$ connecting r, s satisfies the inclusion

$$G(r,s) \subset N. \tag{15}$$

By (14) the geodesic segments $G(p,r), G(p,s)$ are unique. Since $r \neq s$ are endpoints and $r, s \neq p$, an application of (10) yields that

$$G(p,r) \cap G(p,s) = \{p\}. \tag{16}$$

Since $r \notin G(p,s), s \notin G(p,r)$ and $p \notin G(r,s) (\subset N)$ a further application of (10) shows that

$$G(p,r) \cap G(r,s) = \{r\}, \qquad G(p,s) \cap G(r,s) = \{s\}. \tag{17}$$

By (16) and (17) the geodesic segments $G(p,r), G(r,s), G(s,p)$ define a closed Jordan curve on S, say J. Let $U, V \subset S$ be the relatively open Jordan domains determined by J.

(18) For $t \in G(r,s) \setminus \{r,s\}$ the geodesic segment $G(p,t)$ is unique and $G(p,t) \setminus \{p,t\} \subset U$ or $G(p,t) \setminus \{p,t\} \subset V$.

Uniqueness follows from (14) and (15). For the proof of the rest it suffices to show that

$$G(p,t) \cap J = \{p,t\}. \tag{19}$$

By (17) $t \notin G(p,r), G(p,s)$ hold and $r, s \notin G(p,t)$ hold since $r, s \neq p, t$ and both are endpoints. Hence (10) yields that

$$G(p,t) \cap G(p,r) = G(p,t) \cap G(p,s) = \{p\}. \tag{20}$$

From $r, s \notin G(p,t)$ and $p \notin G(r,s) (\subset N)$ together with (10) we infer that

$$G(p,t) \cap G(r,s) = \{t\}. \tag{21}$$

(20) and (21) yield (19) and thus the proof of (18) is concluded.

Let $t(\tau), 0 \leqslant \tau \leqslant \alpha$, be a parametrization of $G(r,s)$ in terms of arc length such that $t(0) = r, t(\alpha) = s$. Define

$$A = \{\tau \in (0,\alpha) \colon G(p, t(\tau)) \subset J \cup U\},$$
$$B = \{\tau \in (0,\alpha) \colon G(p, t(\tau)) \subset J \cup V\}.$$

For the proof that

$$A, B \text{ are closed in } (0, \alpha), \tag{22}$$

it is sufficient to consider the case of A. Let (τ_i) be a sequence in A converging to $\tau_0 \in (0,\alpha)$. We have to prove that $\tau_0 \in A$. By (14) and (15) the geodesic segments $G(p,t(\tau_i)), G(p,t(\tau_0))$ are unique. Since $t(\tau_i) \to t(\tau_0)$, proposition (11) implies that $G(p,t(\tau_i)) \to G(p,t(\tau_0))$. Since $\tau_i \in A$, we obtain that $G(p,t(\tau_i)) \subset J \cup U$. Hence $G(p,t(\tau_0)) \subset J \cup U$ by (12) or, equivalently, $\tau_0 \in A$. This proves (22). Since

$$A \cap B = \emptyset \text{ and } A \cup B = (0,\alpha)$$

by (18) and since the interval $(0,\alpha)$ is connected, (22) can hold only if either $A = \emptyset$ or $B = \emptyset$. We may suppose $B = \emptyset$ and thus $A = (0,\alpha)$. Hence

(23) for all $\tau \in (0,\alpha)$ the unique geodesic segment $G(p,t(\tau))$ is contained in $J \cup U$.

The last step of our proof requires to show that

$$J \cup U = \bigcup \{G(p,t(\tau)) : \tau \in [0,\alpha]\}. \tag{24}$$

Let $u \in J \cup U$. If $u \in J$ then $u \in G(p,r)$ or $u \in G(r,s)$ or $u \in G(s,p)$ and thus $u \in G(p,t(\tau))$ for a suitable $\tau \in [0,\alpha]$. Suppose now that $u \in U$. Let $T(0) = G(p,r)$. For $\tau \in (0,\alpha]$ consider the closed Jordan curve determined by $G(p,r)$, $G(p,t(\tau))$, $G(t(\tau),p)$ (see (17), (20), (21)). By (23) it is contained in $J \cup U$. Let $T(\tau)$ be the closed set bounded by this curve and contained in $J \cup U$ and define

$$\tau_0 = \sup\{\tau \in [0,\alpha] : u \notin T(\tau)\}.$$

Assume first that u is a relative interior point of $T(\tau_0)$. By the definition of τ_0 there is a sequence τ_i in $[0,\alpha]$ with $\tau_i \to \tau_0$ and such that $u \notin T(\tau_i)$. It follows then from (14) and (15) that $G(p,t(\tau_0))$ is unique. Hence (11) yields that $G(p,t(\tau_i)) \to G(p,t(\tau_0))$ Since by assumption u is a relative interior point of $T(\tau_0)$, we thus obtain that u is a relative interior point of $T(\tau_i)$ for i sufficiently large, yielding a contradiction. Assume next that u is an exterior point of $T(\tau_i)$. Since $u \in U \subset T(\alpha)$, we must have $\tau_0 < \alpha$. Since $G(p,t(\tau_0))$ is unique, it follows from (11) that $G(p,t(\tau))$ is arbitrarily close to $G(p,t(\tau_0))$ for $\tau > \tau_0$ sufficiently close to τ_0 in contradiction to the definition of τ_0. Thus the only remaining possibility is that $u \in G(p,t(\tau_0))$, concluding the proof of (24).

(24) contradicts the fact that the relatively open subsect U of S contains endpoints. This proves (13) and thus concludes the proof of Theorem 2.

4. PROOF OF THEOREM 3

Let $d = 3$. As a consequence of the Arzelà-Ascoli theorem the following results, see also [4], p. 75:

(25) Let (G_i) be a sequence of continuous curves of uniformly bounded lengths $\lambda(G_i)$, all contained in a bounded subset of $\mathbb{E}^3$. Then there is a continuous curve G with $G_i \to G$ and $\lambda(G) \leq \liminf_{i\to\infty} \lambda(G_i)$.

(For the concept of convergence of curves see sect. 2.) Arguments of

Alexandrow [1], pp. 377, 378, imply the next two propositions; see also [4], p. 113.

(26) A polytopal convex surface in $\mathbb{E}^3$, for which the sum of the curvatures of any set of vertices never equals 2π, contains no simple closed geodesic.

(27) The set of polytopal convex surfaces in $\mathbb{E}^3$ with the property that the sum of the curvatures of any set of vertices never equals 2π is dense in $\mathcal{S}$.

(The *curvature* of a vertex v of a polytopal convex surface P is three times the volume of the intersection of the solid euclidean unit ball in $\mathbb{E}^3$ with the convex polyhedral cone generated by the exterior normal vectors of the facets of P containing p.)

To prove Theorem 3 define for $k, m = 1, 2, \ldots$,

$\mathcal{S}_{km} = \{S \in \mathcal{S} : S$ contains a simple closed geodesic G with $\lambda(G) \leqslant k$ such that any subarc H of G with $\lambda(H) \leqslant 1/k$ is a geodesic segment and for any pair $p, q \in G$ with distance measured along G at least $1/k$ the inequality $\rho_S(p,q) \geqslant 1/m$ holds$\}$.

From propositions (2), (3) and (25) follows:

$$\mathcal{S}_{km} \text{ is closed in } \mathcal{S}.$$

(26) and (27) imply that

$$\mathcal{S}_{km} \text{ has empty in interior } \mathcal{S}.$$

Thus

$$\bigcup_{k,m} \mathcal{S}_{km} \text{ is meager in } \mathcal{S}.$$

Since this union is precisely the set of all $S \in \mathcal{S}$ that contain a simple closed geodesic, Theorem 3 is proved.

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