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Integral representation and relaxation for functionals defined on measures

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Analisi matematica. — *Integral representation and relaxation for functionals defined on measures.* Nota (*) di ENNIO DE GIORGI, LUIGI AMBROSIO e GIUSEPPE BUTTAZZO, presentata dal Corrisp. E. DE GIORGI.

ABSTRACT. — Given a separable metric locally compact space Ω, and a positive finite non-atomic measure λ on Ω, we study the integral representation on the space of measures with bounded variation Ω of the lower semicontinuous envelope of the functional

$$F(u) = \int_\Omega f(x, u)\, d\lambda \qquad u \in L^1(\Omega, \lambda, \mathbf{R}^n)$$

with respect to the weak convergence of measures.

KEY WORDS: Relaxation; Integral representation; Measures.

RIASSUNTO. — *Rappresentazione integrale e rilassamento per funzionali definiti sulle misure.* Dato uno spazio metrico localmente compatto a base numerabile Ω ed una misura λ su tale spazio, positiva, finita e non atomica, si studia la rappresentazione integrale del funzionale ottenuto rilassando

$$F(u) = \int_\Omega f(x, u)\, d\lambda \qquad u \in L^1(\Omega, \lambda; \mathbf{R}^n)$$

nello spazio $\mathbf{M}_n(\Omega)$ delle misure a variazione limitata su Ω, rispetto alla topologia della convergenza debole di misure.

1. INTRODUCTION

In many problems of Calculus of Variations, given a functional F defined on a topological space (X, τ), it is useful to introduce the so-called (sequentially) τ-*relaxed* functional $\overline{F}$ defined by

$$\overline{F}(x) = \sup\{G(x) : G \text{ is sequentially } \tau\text{-l.s.c.}, G \leq F\}.$$

where $G : X \to \overline{\mathbf{R}}$ is said sequentially τ-l.s.c. if and only if

$$G(x_\infty) \leq \liminf_{h \to +\infty} G(x_h)$$

for every sequence $(x_h) \subset X$ converging to $x_\infty \in X$ in the topology τ.

(*) Pervenuta all'Accademia il 6 agosto 1986.

When F is an integral functional, it is interesting to find an integral representation for the relaxed functional $\overline{F}$. General results of this type have been obtained in the literature either when Ω is a bounded open subset of $\mathbf{R}^k$, X is a Sobolev space $W^{1,p}(\Omega;\mathbf{R}^n)$, τ is the weak $W^{1,p}(\Omega;\mathbf{R}^n)$ topology (or the strong $L^p(\Omega;\mathbf{R}^n)$ topology) and

$$F(u) = \int_{\Omega} f(x, u, Du)\,dx$$

(see for instance [1], [4], [9]), or when X is a space $L^p(\Omega, \lambda;\mathbf{R}^n)$, τ is the weak $L^p(\Omega, \lambda;\mathbf{R}^n)$ topology (or the strong $L^p(\Omega, \lambda;\mathbf{R}^n)$ topology) and

$$F(u) = \int_{\Omega} f(x, u)\,d\lambda(x)$$

where λ is a given measure on a separable locally compact metric space Ω (see for instance [3], [5], [7]).

In this paper we study the τ-relaxation of functionals of the type (see Theorem 2.4)

$$F(\mu) = \begin{cases} \displaystyle\int_{\Omega}^{*} f(x, u)\,d\lambda(x) & \text{if } \mu = u\cdot\lambda \text{ with } u \in L^1(\Omega, \lambda;\mathbf{R}^n) \\ +\infty & \text{otherwise} \end{cases}$$

where μ belongs to the space $\mathbf{M}_n$ of the vector valued measures on Ω with bounded variation, τ is the weak topology of measures, $f:\Omega\times\mathbf{R}^n \to [0, +\infty]$ is a function (not necessarily measurable), and $\int^*$ denotes the upper integral. Similar results, under measurability hypotheses on f, have been obtained with different proofs in [2], [11], [14]. The proof of Theorem 2.4, which guarantees an integral representation for the relaxed functional, is based on an approach rather different from the one followed in [2], [11], [14], and relies on an abstract integral representation theorem for functionals $F(\mu, B)$, depending on measures $\mu \in \mathbf{M}_n$ and Borel sets $B \in \mathbf{B}(\Omega)$ (see Theorem 2.3).

The integral representation theorem given in this paper is the natural generalization of the theorem given in [3] concerning functionals $F(u, B)$, depending on $u \in L^1(\Omega, \lambda;\mathbf{R}^n)$ and $B \in \mathbf{B}(\Omega)$. Finally, in the last section of the paper we show some examples for which it is possible to compute explicitly the relaxed functional.

2. Statement of the results.

In this section $(\Omega, \mathbf{B}, \lambda)$ will denote a measure space, where Ω is a separable metric locally compact space, $\mathbf{B}$ is the σ-algebra of the Borel subsets of Ω, and $\lambda:\mathbf{B}\to[0, +\infty[$ is a positive, non-atomic, finite measure.

For every vector measure $\mu : \mathbf{B} \to \mathbf{R}^n$ and every $B \in \mathbf{B}$ the variation of μ on B is defined by

$$|\mu|(B) = \sup\left\{\sum_{h=1}^{\infty} |\mu(B_h)| : B_h \in \mathbf{B}, \bigcup_{h=1}^{\infty} B_h \subset B, B_h \text{ pairwise disjoint}\right\},$$

We consider the following spaces:

$\mathbf{M}_n$ the space of all vector measures $\mu : \mathbf{B} \to \mathbf{R}^n$ with finite variation on Ω;

L^p_n the space of all λ-measurable functions $u : \Omega \to \mathbf{R}^n$ with $\int_\Omega |u|^p d\lambda < +\infty$;

C^0_n the space of all continuous functions $u : \Omega \to \mathbf{R}^n$ "vanishing on the boundary", that is for every $\varepsilon > 0$ there exists a compact subset K_ε of Ω such that $|u(x)| < \varepsilon$ for all $x \in \Omega - K_\varepsilon$.

The space $\mathbf{M}_n$ can be identified with the dual space of C^0_n by the duality (see [13], page 40)

$$\langle\mu, u\rangle_\Omega = \sum_{i=1}^{n} \int_\Omega u^i \, d\mu_i \qquad (u \in C^0_n, \mu \in \mathbf{M}_n),$$

so that a sequence (μ_h) in $\mathbf{M}_n$ is weak*-convergent to $\mu \in \mathbf{M}_n$ if and only if

$$\langle\mu_h, u\rangle_\Omega \to \langle\mu, u\rangle_\Omega \qquad \text{for every } u \in C^0_n.$$

In the following, given $u \in L^1_n$, we denote by $u \cdot \lambda$ the measure of $\mathbf{M}_n$ defined by

$$(u \cdot \lambda)(B) = \int_B u \, d\lambda \qquad \text{for every } B \in \mathbf{B}.$$

DEFINITION 2.1. *We say that* $\mu \in \mathbf{M}_n$ *is* absolutely continuous *with respect to* λ *(and we write* $\mu \ll \lambda$*) if*

$$|\mu|(B) = 0 \qquad \textit{whenever} \quad B \in \mathbf{B} \quad \textit{and} \quad \lambda(B) = 0.$$

We say that $\mu \in \mathbf{M}_n$ *is* singular *with respect to* λ *(and we write* $\mu \perp \lambda$*) if*

$$|\mu|(\Omega - B) = 0 \qquad \textit{for a suitable} \quad B \in \mathbf{B} \qquad \textit{with} \quad \lambda(B) = 0.$$

It is well-known that every absolutely continuous measure $\mu \in \mathbf{M}_n$ is representable in the form $\mu = a \cdot \lambda$ for a suitable $a \in L^1_n$; moreover, the following Lebesgue-Nykodim decomposition result for measures of $\mathbf{M}_n$ holds (see [13] page 122).

PROPOSITION 2.2. *For every* $\mu \in \mathbf{M}_n$ *there exist a unique function* $a \in L_n^1$ *and a unique measure* $\mu^s \in \mathbf{M}_n$ *such that*

i) $\mu = a \cdot \lambda + \mu^s$;

ii) μ^s *is singular with respect to* λ.

The function a *is often indicated by* $\dfrac{d\mu}{d\lambda}$.

For proper convex functions $f : \mathbf{R}^n \to]-\infty, +\infty]$ we define as usual the recession function of f (see [12]) by

$$f^\infty(s) = \lim_{t \to +\infty} \frac{f(w + ts)}{t} \qquad \text{for every } s \in \mathbf{R}^n,$$

where w is any point in $\mathbf{R}^n$ such that $f(w) < +\infty$; in fact, the definition above is actually independent of the choice of w.

We are now in a position to state our integral representation result.

THEOREM 2.3. *Let* $\Phi : \mathbf{M}_n \times \mathbf{B} \to]-\infty, +\infty]$ *be a functional satisfying the following properties:*

i) Φ *is* $\mathbf{B}$*-local (that is* $\Phi(\mu, B) = \Phi(\nu, B)$ *whenever* $\mu, \nu \in \mathbf{M}_n$, $B \in \mathbf{B}$, *and* $|\mu - \nu|(B) = 0$);

ii) *for every* $\mu \in \mathbf{M}_n$ *the set function* $\Phi(\mu, \cdot)$ *is finitely additive;*

iii) *the functional* $\Phi(\cdot, \Omega)$ *is convex and sequentially lower semicontinuous with respect to the weak*-convergence on* $\mathbf{M}_n$;

iv) *there exists* $u_0 \in L_n^1$ *such that* $\Phi(u_0, B) < +\infty$ *for every* $B \in \mathbf{B}$;

v) *for every* $\mu \in \mathbf{M}_n$ *singular with respect to* λ *the function* $t \to \Phi(u_0 + t\mu, \Omega) - \Phi(u_0, \Omega)$ *is positively* 1*-homogeneous.*

Then, there exists a Borel function $\phi : \Omega \times \mathbf{R}^n \to]-\infty, +\infty]$ *such that*

a) for every $x \in \Omega$ *the function* $\phi(x, \cdot)$ *is convex and lower semicontinuous on* $\mathbf{R}^n$;

b) there exist $a \in L_1^1$ *and* $b \geq 0$ *such that* $\phi(x, s) \geq -b|s| + a(x)$ *for* λ-a.e. $x \in \Omega$ *and for all* $s \in \mathbf{R}^n$;

c) the following integral representation formula holds for every $\mu \in \mathbf{M}_n$:

$$\Phi(\mu, B) - \Phi(u_0 B) = \int_B \phi\left(x, \frac{d\mu}{d\lambda}\right) d\lambda + \int_B \phi^\infty\left(x, \frac{d\mu^s}{d|\mu|}\right) d|\mu|$$

where $\mu = \frac{d\mu}{d\lambda} \cdot \lambda + \mu^s$ *is the Lebesgue-Nykodim decomposition of* μ, *and for every* $x \in \Omega$ $\phi^\infty(x, \cdot)$ *is the recession function of* $\phi(x, \cdot)$.

d) *the function* $\Phi^\infty(x, s)$ *is lower semicontinuous in* (x, s).

By using the integral representation theorem above, we can solve the relaxation problem which can be stated as follows. Let $f : \Omega \times \mathbf{R}^n \to [0, +\infty]$ be a given function (note that no measurability hypotheses are required); for every $\mu \in \mathbf{M}_n$ define

$$
\text{(2.1)} \qquad F(\mu) = \begin{cases} \displaystyle\int_\Omega^* f(x, u(x))\, d\lambda(x) & \text{if } \mu = u \cdot \lambda \text{ with } u \in L^1_n \\ +\infty & \text{otherwise,} \end{cases}
$$

where $\int^*$ denotes the upper integral. We are interested in the characterization of the greatest functional Φ on $\mathbf{M}_n$ which is sequentially w^*-l.s.c. and less than or equal to F; in particular, we want to write Φ in the form

$$
\text{(2.2)} \qquad \Phi(\mu) = \int_\Omega \phi\left(x, \frac{d\mu}{d\lambda}\right) d\lambda(x) + \int_\Omega \phi^\infty\left(x, \frac{d\mu^s}{d|\mu|}\right) d|\mu|
$$

for a suitable integrand ϕ. The following result holds.

Theorem 2.4. *Assume that the functional* F *defined in* (2.1) *is finite in at least one* $u_0 \in L^1_n$. *Then there exists a Borel function* $\phi(x, s)$, *convex and lower semicontinuous in* s, *such that* (2.2) *holds for every* $\mu \in \mathbf{M}_n$. *Moreover,* $\phi^\infty(x, s)$ *is lower semicontinuous in* (x, s).

3. Some examples

In [10] Olech found a characterization of all integrands ϕ such that the functional

$$
\Phi(u) = \int_\Omega \phi(x, u)\, d\lambda
$$

is sequentially w^*-$\mathbf{M}_n$ lower semicontinuous on the space L^1_n. His result is that Φ is sequentially w^*-$\mathbf{M}_n$ lower semicontinuous if and only if there exist a sequence of functions $a_h \in L^1_n$ and a sequence of functions $b_h \in C^0_n$ such that

$$
\phi(x, u) = \sup\{a_h(x) + \langle b_h(x), u\rangle : h \in \mathbf{N}\} \qquad \forall\, u \in \mathbf{R}^n
$$

for λ-*a.e.* $x \in \Omega$. By using this result, it is possible to find an explicit characterization of the integrand given by Theorem 2.4 in some interesting cases (see also for instance [6], [8]).

EXAMPLE 1. Let $f : \Omega \times \mathbf{R}^n \to \mathbf{R}$ be the function

$$f(x, s) = a(x) \, |s|$$

where $a : \Omega \to [0, +\infty]$ is a measurable function. Then the relaxed functional Φ of Theorem 2.4 can be represented in the form (2.2) with ϕ given by

$$\phi(x, s) = \tilde{a}(x) \, |s|$$

where $\tilde{a}$ is the greatest lower semicontinuous function on Ω less than or equal to a almost everywhere on Ω. If $a \in L^1_{loc}(\Omega)$, it is easy to see that the following formula holds

$$\tilde{a}(x) = \liminf_{y \to x} \limsup_{\rho \to 0} \fint_{B_\rho(y)} a(t) \, dt \qquad \text{for every } x \in \Omega. \tag{3.1}$$

EXAMPLE 2. Let $a \in L^1_1$ and let $f : \Omega \times \mathbf{R}^n \to \mathbf{R}$ be the function

$$f(x, s) = a(x) \sqrt{1 + |s|^2}.$$

If we denote by $\tilde{a}$ the function defined in (3.1), then the relaxed functiona Φ given by Theorem 2.4 can be represented in the form (2.2) with ϕ given by

$$\phi(x, s) = \begin{cases} \tilde{a}(x) \sqrt{1 + |s|^2} & \text{if } a(x) \le \tilde{a}(x) \\ a(x) \sqrt{1 + |s|^2} & \text{if } a(x) > \tilde{a}(x) \text{ and } |s| \le \\ & \le \dfrac{\tilde{a}(x)}{\sqrt{a^2(x) - \tilde{a}^2(x)}} \\ \tilde{a}(x) \, |s| + \sqrt{a^2(x) - \tilde{a}^2(x)} & \text{otherwise.} \end{cases}$$

EXAMPLE 3. Let $a : \Omega \to [0, +\infty]$ be a measurable function and let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be the function

$$f(x, s) = a(x) \, |s|^p \qquad (\text{with } p > 1).$$

Then, the relaxed functional Φ given by Theorem 2.4 can be represented in the form (2.2) with ϕ given by

$$\phi(x, s) = a^*(x) \, |s|^p$$

where a^* is the function

$$a^*(x) = \begin{cases} 0 & \text{if } x \in \Omega - \mathrm{U} \\ +\infty & \text{if } x \in \mathrm{U} \text{ and } a(x) = 0 \\ a(x) & \text{otherwise.} \end{cases}$$

and U is the greatest open subset of Ω such that $a^{1/(1-p)} \in L^1_{loc}(\mathrm{U})$.

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