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On the Convergence of the Approximate Free Boundary for the Parabolic Obstacle Problem

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Analisi numerica. — *On the Convergence of the Approximate Free Boundary for the Parabolic Obstacle Problem.* Nota di PAOLA PIETRA (*) e CLAUDIO VERDI (**), presentata (***) dal Corrisp. E. MAGENES.

RIASSUNTO. — Si discretizza il problema dell'ostacolo parabolico con differenze all'indietro nel tempo ed elementi finiti lineari nello spazio e si dimostrano stime dell'errore per la frontiera libera discreta.

0. INTRODUCTION

An important question in dealing with the numerical solution of free boundary problems is the approximation of the free boundary itself. The feature of the problem does not allow general conclusions, since the discrete free boundary could be a set with no relation to the continuous free boundary. If non-degeneracy properties of the solutions are known, results as to the accuracy in the approximation of the free boundary can be given (see Brezzi and Caffarelli [1]; Nochetto [10]; Pietra and Verdi [11]).

The aim of this paper is to analyse the behaviour of the discrete free boundary for a parabolic obstacle problem, discretized with backward-differences in time and linear finite elements in space. The continuous solution does satisfy non-degeneracy properties and the discrete one reproduces the same behaviour, provided the decomposition is of *acute-type*. So the discrete free boundary is allowed to be defined in the natural way as the boundary of the contact set and a rate of convergence to the continuous free boundary can be proved. The measure of the symmetric-difference of the continuous and discrete coincidence sets or the distance between the free boundaries is bounded in terms of the L^∞-error estimate for the solutions.

An outline of the paper is as follows.

In Section 1 the non-degeneracy properties of the solution of the continuous problem are proved. The discrete problem is stated in Section 2. Section 3 deals with the non-degeneracy property of the discrete solution and with error estimates between the continuous and discrete free boundaries.

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Let us introduce some notation.

μ : Lebesgue measure in $\mathbf{R}^{N+1}$;

$d((x, t), (\bar{x}, \bar{t})) = \max(|x - \bar{x}|, |t - \bar{t}|^{1/2})$;

$Q^\varepsilon(\bar{x}, \bar{t}) = \{(x, t) \in \mathbf{R}^{N+1} : d((x, t), (\bar{x}, \bar{t})) < \varepsilon\}$;

$Q_\varepsilon(\bar{x}, \bar{t}) = \{(x, t) \in Q^\varepsilon(\bar{x}, \bar{t}), t < \bar{t}\}$;

$\mathbf{S}_\varepsilon(E) = \{(x, t) \in \mathbf{R}^{N+1}: d((x, t), E) < \varepsilon\}$, $\forall E \subset \mathbf{R}^{N+1}$, $\forall \varepsilon > 0$.

1. The continuous problem

Let Ω be an open bounded set of $\mathbf{R}^N$ $(N \geq 1)$. We set $Q = \Omega \times]0, T[$, $T < \infty$; $\partial_p Q = (\partial\Omega \times]0, T[) \cup (\Omega \times \{0\})$. For any $\delta > 0$, we set $\Omega_{\{-\delta\}} = \{x \in \Omega: d(x, \partial\Omega) \geq \delta\}$, $Q_{\{-\delta\}} = \Omega_{\{-\delta\}} \times [\delta^2, T[$, and for any set $E \subset \mathbf{R}^N$ $(E \subset \mathbf{R}^{N+1})$, we define $E^\delta = E \cap \Omega_{\{-\delta\}}$ $(E^\delta = E \cap Q_{\{-\delta\}})$.

Given functions $f \in L^2(Q)$ and $g \in H^1(Q)$, with $g \geq 0$ on $\partial_p Q$, we define the convex set

$$W = \{v \in H^1(Q), v \geq 0 \text{ a.e. in } Q, v = g \text{ on } \partial_p Q\} \tag{1}$$

and consider the following problem

To find $u \in W$ such that

$$\text{(P)} \quad \int_\Omega u_t (v - u)\, dx + \int_\Omega \nabla u \cdot \nabla (v - u)\, dx \geq \int_\Omega f(v - u)\, dx \quad \forall v \in W, \text{ a.e. } t \in]0, T[.$$

It is well known (see, e.g., Friedman [8]) that if $\partial\Omega \in C^{2+\alpha}$ and

$$f, g, g_t, D_x g, D_x^2 g \quad \text{belong to} \quad C^\alpha(\bar{Q}), 0 < \alpha < 1 \tag{2}$$

then the problem (P) has one and only one solution satisfying

$$u_t, D_x u, D_x^2 u \quad \text{belong to} \quad L^\infty_{loc}(Q). \tag{3}$$

Moreover,

$$\text{if } f \in H^1(Q) \quad \text{then} \quad u_t \in L^2(0, T; H^1(\Omega)). \tag{4}$$

Let us define the positivity set and the free boundary

$$P = \{(x, t) \in Q : u(x, t) > 0\}, \qquad \Gamma = \partial P \cap Q. \tag{5}$$

From now on, C, C_1, etc. will denote constants independent of the point (x, t) and ε. Finally, set

$$\partial_p^+ Q = \{(x, t) \in \partial_p Q : g(x, t) > 0\} .$$

Throughout the paper we shall assume assumptions (2) and $f \in H^1(Q)$. Under the further hypothesis

$$-f(x, t) \geq \lambda > 0 \quad \text{in } Q \tag{6}$$

we shall prove some properties of the free boundary and non-degeneracy properties of the solution of the problem (**P**).

The following Lemma 1 shows that the solution u cannot be uniformly small in some neighbourhood of a point on the free boundary.

LEMMA 1. *Let* $(\bar{x}, \bar{t})$ *be any point in* $\bar{P}$. *For any cylinder* $Q_\varepsilon(\bar{x}, \bar{t})$ *with* $Q_\varepsilon(\bar{x}, \bar{t}) \cap \partial_p^+ Q = \phi$, *we have*

$$\sup_{(x, t) \in Q_\varepsilon(\bar{x}, \bar{t}) \cap P} u(x, t) - u(\bar{x}, \bar{t}) \geq \frac{\lambda}{2N+1} \varepsilon^2 . \tag{7}$$

Proof. Suppose first that $(\bar{x}, \bar{t}) \in P$. The function

$$w(x, t) = u(x, t) - u(\bar{x}, \bar{t}) - \frac{\lambda}{2N+1} (|x - \bar{x}|^2 + |t - \bar{t}|) \tag{8}$$

satisfies $w(\bar{x}, \bar{t}) = 0$ and $\Delta w - w_t \geq 0$ in P. Hence, by the parabolic maximum principle, sup w in $\overline{P \cap Q_\varepsilon(\bar{x}, \bar{t})}$ is non-negative and it is attained on $(\Gamma \cap Q_\varepsilon(\bar{x}, \bar{t})) \cup (\partial_p Q_\varepsilon(\bar{x}, \bar{t}) \cap \bar{P})$. Since $w(x, t) < 0$ if $(x, t) \in \Gamma$ and $Q_\varepsilon(\bar{x}, \bar{t}) \cap \partial_p^+ Q = \phi$, actually there must exist a point $(y, \tau) \in \partial_p Q_\varepsilon(\bar{x}, \bar{t}) \cap \bar{P}$, such that $w(y, \tau) \geq 0$, hence the thesis follows.

If $(\bar{x}, \bar{t}) \notin P$, we apply the result to a sequence of points $(x_n, t_n) \in P$, with $(x_n, t_n) \to (\bar{x}, \bar{t})$.

■

In order to prove that the Lebesgue measure of Γ is equal to zero (see Theorem 1) we need the following Lemma 2.

LEMMA 2. *For any* $\delta > 0$, *there exist two positive constants* ε_0 *and* γ *such that, for any point* $(\bar{x}, \bar{t}) \in \Gamma^\delta$, *we have*

$$\frac{\mu(Q_\varepsilon(\bar{x}, \bar{t}) \cap P)}{\mu(Q_\varepsilon(\bar{x}, \bar{t}))} \geq \gamma, \quad \forall\, 0 < \varepsilon \leq \varepsilon_0 . \tag{9}$$

Proof. By Lemma 1, there exists a point $(y, \tau) \in \overline{Q_\varepsilon(\bar{x}, \bar{t})}$ so that $u(y, \tau) = C_1 \varepsilon^2$. Since $u \geq 0$ and $u \in L^\infty_{loc}(0, T, C^{1,1}_{loc}(\Omega))$, by Remark 1 in Caffarelli [2] we have $|\nabla u(y, \tau)| \leq C_2 \varepsilon$ and then $C_3 \varepsilon^2 \leq u(x, \tau) \leq C_4 \varepsilon^2$, $\forall\, |x - y| < C_5 \varepsilon$. Since $u_t \in L^\infty_{loc}(Q)$, then $C_6 \varepsilon^2 \leq u(x, t) \leq C_7 \varepsilon^2$, $\forall\, (x, t) \in Q_{C\varepsilon}(y, \tau)$, hence the lemma follows.

■

Remark 1. Setting

$$A_\varepsilon = \{(x, t) \in Q : 0 < u(x, t) \leq \varepsilon^2\}, \tag{10}$$

properties (3) entails the following Lemma 3.

LEMMA 3. *For any* $\delta > 0$, *there exist positive constants* ε_0, C *and* γ *such that, for any point* $(\bar{x}, \bar{t}) \in \Gamma^\delta$, *we have*

$$\frac{\mu(Q_\varepsilon(\bar{x}, \bar{t}) \cap A^{\delta/2}_{C\varepsilon})}{\mu(Q_\varepsilon(\bar{x}, \bar{t}))} \geq \gamma, \qquad \forall\, 0 < \varepsilon \leq \varepsilon_0. \tag{11}$$

■

THEOREM 1. *The free boundary* Γ *has Lebesgue measure zero.*

Proof. Lemma 2 implies that

$$\lim_{\varepsilon \to 0} \frac{\mu(Q^\varepsilon(\bar{x}, \bar{t}) \cap \Gamma)}{\mu(Q^\varepsilon(\bar{x}, \bar{t}))} < 1 \quad \forall\, (\bar{x}, \bar{t}) \in \Gamma. \tag{12}$$

For any measurable set $E \subset \mathbf{R}^{N+1}$, we have (see, e.g., Federer [7] Sec. 2.9)

$$\lim_{\varepsilon \to 0} \frac{\mu(Q^\varepsilon(\bar{x}, \bar{t}) \cap E)}{\mu(Q^\varepsilon(\bar{x}, \bar{t}))} = 1 \quad \text{a.e. } (\bar{x}, \bar{t}) \in E. \tag{13}$$

Since Γ is measurable, then (12) and (13) imply $\mu(\Gamma) = 0$.

■

It can be proved that the free boundary Γ has a N–dimensional Hausdorff measure finite. To this end we prove the following theorem.

THEOREM 2. *For any* $\delta > 0$, *there exist two positive constants* ε_0 *and* C *such that*

$$\mu(A^\delta_\varepsilon) \leq C\varepsilon, \qquad \forall\, 0 < \varepsilon \leq \varepsilon_0. \tag{14}$$

In order to prove Theorem 2, we need two lemmata.

LEMMA 4. *For any $\delta > 0$, there exist two positive constants ε_0 and* C *such that*

$$\int_{A_\varepsilon^\delta} (\Delta u)^2 \, dx \, dt \leq C \, \varepsilon \,, \qquad \forall\, 0 < \varepsilon \leq \varepsilon_0 \,. \tag{15}$$

Proof. The proof is carried out with techniques used in Caffarelli [2] for an analogous estimate in the elliptic case. Let

$$\mathbf{O}_\varepsilon = \{(x\,,t) \in P : |\nabla u(x\,,t)| < \varepsilon\}\,,\ \mathbf{O}_\varepsilon^i = \{(x\,,t) \in P : |D_i u(x\,,t)| < \varepsilon\}, \tag{16}$$

and set $h_\varepsilon^i(x\,,t) = (D_i u(x\,,t) \wedge \varepsilon) \vee (-\varepsilon)$. We note that $(\Delta u)^2 \leq N \sum_i (D_{ii} u)^2$, hence we get

$$\int_{O_\varepsilon^\delta} (\Delta u)^2 \, dx \, dt \leq N \int_{O_\varepsilon^\delta} \sum_i (D_{ii} u)^2 \, dx \, dt \leq N \sum_i \int_{(O_\varepsilon^i)^\delta} (D_{ii} u)^2 \, dx \, dt \,, \tag{17}$$

since $\mathbf{O}_\varepsilon \subset \mathbf{O}_\varepsilon^i$. Moreover

$$\int_{(O_\varepsilon^i)^\delta} (D_{ii} u)^2 \, dx \, dt \leq \int_{(O_\varepsilon^i)^\delta} \nabla D_i u \cdot \nabla D_i u \, dx \, dt = \int_{Q_{\{-\delta\}}} \nabla h_\varepsilon^i \cdot \nabla D_i u \, dx \, dt \,. \tag{18}$$

Noting that $\mu(\Gamma) = 0$ (see Theorem 1), $h_\varepsilon^i = 0$ on Γ, $h_\varepsilon^i \in L^\infty_{\text{loc}}(0\,,T;$ $C^{0,1}_{\text{loc}}(\Omega))$ and $-u_t + f = -\Delta u$ in P^δ, we obtain (recall (4))

$$\begin{aligned} \int_{Q_{\{-\delta\}}} \nabla h_\varepsilon^i \cdot \nabla D_i u \, dx \, dt &= \int_{P^\delta} (D_i(-u_t + f)) \, h_\varepsilon^i \, dx \, dt \, + \\ &\int_{\delta^2}^{T} \int_{\partial\Omega_{\{-\delta\}} \cap P(t)} (\partial_\nu D_i u) \, h_\varepsilon^i \, ds \, dt \end{aligned} \tag{19}$$

by using approximation arguments. By Remark 1 in Caffarelli [2]

$$A_\varepsilon^\delta \subset \mathbf{O}_{C\varepsilon}^\delta \,, \tag{20}$$

hence from (20), (17), (18), (19) the assertion (15) follows. ∎

LEMMA 5. *For any $\delta > 0$, there exist two positive constants ε_0 and* C *such that*

$$\int_{A_\varepsilon^\delta} f \, u_t \, dx \, dt \leq C \varepsilon^2 \,, \quad \forall\, 0 < \varepsilon \leq \varepsilon_0 \,. \tag{21}$$

Proof. Let $B_\varepsilon(x) = \{t \in]0, T[: (x, t) \in A_\varepsilon\}$ and set $u_\varepsilon(x, t) = u(x, t) \wedge \wedge \varepsilon^2$. By using Fubini theorem and integrating by parts, we obtain

$$\begin{aligned}(22)\qquad \int_{A_\varepsilon^\delta} f\, u_t \,dx\,dt &= \int_{\Omega_{]-\delta]}} \int_{B_\varepsilon(x)\cap]\delta^2, T[} f\, u_t \,dt\,dx = \int_{Q_{]-\delta]}} f\, u_{\varepsilon,t}\,dt\,dx = \\ &= -\int_{Q_{]-\delta[}} f_t\, u_\varepsilon\,dt\,dx + \int_{\Omega_{]-\delta}} f u_\varepsilon \,\Big|_{\delta^2}^{T} \le C\,\varepsilon^2 ,\end{aligned}$$

because $f \in H^1(Q)$.

■

Proof of Theorem 2. In the set P we have $\Delta u - u_t = -f \ge \lambda > 0$, so that

$$(23)\qquad \lambda^2 \le (\Delta u - u_t)^2 = (\Delta u)^2 - u_t^2 + 2\, u_t f \le (\Delta u)^2 + 2\, u_t f .$$

Then we obtain

$$(24)\qquad \mu(A_\varepsilon^\delta) = \frac{1}{\lambda^2} \int_{A_\varepsilon^\delta} \lambda^2 \,dx\,dt \le \frac{1}{\lambda^2} \int_{A_\varepsilon^\delta} ((\Delta u)^2 + 2\, u_t f)\,dx\,dt ,$$

hence Lemma 4 and Lemma 5 yield the thesis.

■

The strip $\mathbf{S}_\varepsilon(\Gamma^\delta)$ can be obtained by covering Γ^δ by means of cylinders $Q^{C_1\varepsilon}(\bar{x}, \bar{t})$, $(\bar{x}, \bar{t}) \in \Gamma^\delta$. One may restrict the coverings to be such that each point of Γ^δ is contained only in a finite number N^* of cylinders, where N^* is independent of the covering and of ε.

Theorem 3. *For any $\delta > 0$, there exist two positive constants ε_0 and C such that*

$$(25)\qquad \mu(\mathbf{S}_\varepsilon(\Gamma^\delta)) \le C\varepsilon , \qquad \forall\, 0 < \varepsilon \le \varepsilon_0 .$$

Proof. By Lemma 3, there exist positive constants ε_0, C_2 and γ such that

$$(26)\qquad \mu(\mathbf{S}_\varepsilon(\Gamma^\delta)) \le \Sigma\, \mu(Q^{C_1\varepsilon}) \le \frac{2}{\gamma}\, \Sigma \mu(Q^{C_1\varepsilon} \cap A_{C_2\varepsilon}^{\delta/2}) \le \frac{N^*}{\gamma}\, \mu(A_{C_2\varepsilon}^{\delta/2}) ,$$

hence Theorem 2 yields the property (25).

■

2. THE DISCRETE PROBLEM

The problem P is discretized by piecewise linear finite elements in space and by backward differences in time.

Let $m \geq 1$ be a integer, $k = T/m$ be the time-step and $t^i = i\,k$, $i = 0, \ldots, m$. Let $\{\mathcal{T}_h\}_h$ be a family regular and quasi-uniform of decompositions of Ω into closed N-simplices (see, e.g., Ciarlet [4]); h stands for the mesh-size. For the sake of convenience, suppose that $\overline{\Omega} = \Omega_h = \bigcup_{\tau \in \mathcal{T}_h} \tau$, that is, we are considering polygonal domains. Let now $\{x_j\}_{j=1}^n$ be the nodes of $\mathcal{T}_h$, numbered as follows

$$\text{(27)} \qquad \begin{aligned} &\{x_j\}_{j=1}^{n_0} \text{ are the internal nodes,} \\ &\{x_j\}_{j=n_0+1}^{n} \text{ are the nodes on the boundary } \partial\Omega, \end{aligned}$$

and denote by $\mathbf{W}$ ($\widehat{\mathbf{W}}$) the vector of $\mathbf{R}^{n_0}$ ($\mathbf{R}^n$) of components W_j, $j = 1, \ldots, n_0$ ($j = 1, \ldots, n$). Let us set

$$\text{(28)} \qquad \mathbf{V}_h = \{\chi \in C^0(\overline{\Omega}) : \chi_{|\tau} \text{ is linear } \forall\ \tau \in \mathcal{T}_h\}$$

and indicate by $\{\varphi_j\}_{j=1}^n$ the canonical basis of $\mathbf{V}_h$.

The integrals $\int_\Omega \varphi_j\,\varphi_l\,dx$ will be calculated by the following quadrature formula

$$\text{(29)} \qquad (\varphi_j, \varphi_l)_h = 0 \text{ if } j \neq l, \qquad (\varphi_j, \varphi_j)_h = \int_\Omega \varphi_j\,dx\,.$$

Let us define the matrices

$$\text{(30)} \qquad \begin{aligned} \mathbf{M} &= \{m_{jl}\} = \{(\varphi_j, \varphi_l)_h\}_{j,l=1}^{n_0}, \\ \widehat{\mathbf{A}} &= \{a_{jl}\} = \{\int_\Omega \nabla\varphi_j \cdot \nabla\varphi_l\,dx\}_{j=1,l=1}^{n_0,n}, \qquad \mathrm{A} = \{a_{jl}\}_{j,l=1}^{n_0}, \\ \mathbf{B} &= \{b_{jl}\} = \mathbf{M} + k\,\mathbf{A}\,. \end{aligned}$$

Setting

$$\text{(31)} \qquad \begin{aligned} F_j^i &= \int_\Omega f(x, t^i)\,\varphi_j(x)\,dx\,, \\ G_j^i &= \sum_{l=n_0+1}^{n} a_{jl}\; g(x_l, t^i)\,, \end{aligned} \qquad j = 1, \ldots, n_0,\ i = 1, \ldots, m,$$

the discrete problem is the following

$$(\mathrm{P}_{h,k}) \qquad u_{hk}^0 = \sum_{j=1}^{n} g(x_j, 0)\,\varphi_j$$

$$\textit{for } i = 1, \ldots, m, \textit{ find } u_{hk}^i = \sum_{j=1}^{n} \mathrm{U}_j^i \varphi_j \in \mathrm{V}_h \textit{ such that}$$

$$\mathrm{U}_j^i = g(x_j, t^i), \qquad j = n_0 + 1, \ldots, n,$$

$$(32\mathrm{i}) \qquad \mathbf{U}^i \geq \mathbf{0}$$

$$(32\mathrm{ii}) \qquad \mathbf{B}\,\mathbf{U}^i \geq k(\mathbf{F}^i - \mathbf{G}^i) + \mathbf{M}\,\mathbf{U}^{i-1}$$

$$(32\mathrm{iii}) \qquad (\mathbf{B}\,\mathbf{U}^i - k(\mathbf{F}^i - \mathbf{G}^i) - \mathbf{M}\,\mathbf{U}^{i-1})\,\mathbf{U}^i = 0\,.$$

This problem has one and only one solution (see, e.g., Glowinski, Lions and Trémolières [9]). The discrete solution of the problem (P) is the function

$$u_{hk}(\cdot, t) = u_{hk}^i(\cdot) \quad \text{if } t \in]t^{i-1}, t^i]\,.$$

The assumptions on the finite element space imply

$$(33) \qquad \sum_{l=1}^{n} a_{jl} = 0\,, \qquad j = 1, \ldots, n_0\,.$$

In order to obtain a parabolic discrete maximum principle (P.D.M.P.) we need further hypotheses on the decomposition. We consider decompositions of *acute-type* (see, e.g., Ciarlet [3], Ciarlet and Raviart [5]). In these cases

$$(34) \qquad a_{jl} \leq 0 \qquad \text{if } j \neq l\,.$$

Denote by $\tau^i = \tau \times [t^{i-1}, t^i]$, $\tau \in \mathcal{T}_h$, $i = 1, \ldots, m$, and set $\mathcal{T}_{hk} = \{\tau^i : \tau \in \mathcal{T}_h;\ 1 \leq i \leq m\}$. Consider a connected union of elements of $\mathcal{T}_{hk}$. We shall denote by D either this set or the set of its nodes and define

$$\partial_p \mathrm{D} = \{(x_j, t^i) \in \mathrm{D} : (x_j, t^{i-1}) \notin \mathrm{D} \ \text{ or } \ \exists x_l \in \operatorname{supp} \varphi_j : (x_l, t^i) \notin \mathrm{D}\}\,.$$

We report the statement of the P.D.M.P. The proof is an easy extension of that one in the elliptic case.

THEOREM 4 (P.D.M.P.). *Let* D *be a connected union of elements of* $\mathcal{T}_{hk}$. *Let* $\{\widehat{\mathbf{W}}^i\}_{i=0}^m \in \mathbf{R}^{n\times(m+1)}$ *such that*

$$(35) \qquad m_{jj}(\mathrm{W}_j^i - \mathrm{W}_j^{i-1}) + k(\hat{\mathbf{A}}\hat{\mathbf{W}}^i)_j < 0\,, \qquad \forall (x_j, t^i) \in \mathrm{D} - \partial_p \mathrm{D}\,.$$

Then

(36)
$$\max_{(x_j,t^i)\in\partial_p D} W_j^i > \max_{(x_j,t^i)\in D-\partial_p D} W_j^i .$$

3. APPROXIMATION OF THE FREE BOUNDARY

We define the discrete positivity set and free boundary

(37)
$$P_{hk} = \{(x\,,t\} \in Q : u_{hk}(x\,,t) > 0\}\,, \qquad \Gamma_{hk} = \partial P_{hk} \cap Q\,,$$

and observe that $\overline{P_{hk}}$ is an union of elements of $\mathcal{T}_{hk}$.

The following Theorem 5 is a result of non-degeneracy for the discrete solution.

THEOREM 5. *There exist three positive constants γ_0, h_0 and k_0 such that, for any $0 < h \leq h_0$, for any $0 < k \leq k_0$, for any $r \geq 2h$ and $r^2 \geq \frac{4}{3}\,k$ and for any node $(x_*\,,t^*) \in P_{hk}$ with $Q_r(x_*\,,t^*) \cap \partial_p^+ Q = \phi$, we have*

(38)
$$\overline{\max_{(x_j,t^i)\in Q_r(x_*,t^*)\cap P_{hk}}} U_j^i > \gamma_0\, r^2 .$$

Proof. Set $\sigma^*(x) = |\,x - x_*\,|^2$ and consider the vector $\hat{\Theta}^*$ of its nodal values. By Theorem 3.3 of Brezzi and Caffarelli [1] there exist two positive constants δ_0 and δ_1, independent of h and x_*, such that

(39)
$$-\,\delta_0 \int_\Omega \varphi_j(x)\,dx \leq (\hat{A}\hat{\Theta}^*)_j \leq -\,\delta_1 \int_\Omega \varphi_j(x)\,dx\,, \qquad j = 1\,,\ldots,n_0\,.$$

For $i = 0\,,\ldots,m$, consider the vector $\widehat{\mathbf{W}}^i$ of components

(40)
$$W_j^i = U_j^i - \frac{\lambda}{2\,(\delta_0 + 1)}\,(\sigma^*(x_j) + |\,t^i - t^*\,|)\,, \qquad j = 1\,,\ldots,n\,,$$

where λ is the constant defined in (6).

Let D the connected biggest union of elements of $\mathcal{T}_{hk}$ such that $D - \partial_p D \subset P_{hk} \cap Q_r(x_*\,,t^*)$ and such that $x_* \in \operatorname{int}\{x : (x\,,t^*) \in D\}$. By (32.iii), in the nodes $(x_j\,,t^i) \in D - \partial_p D$, we have

(41)
$$m_{ij}\,(U_j^i - U_j^{i-1}) + k\,(\widehat{\mathbf{A}}\,\widehat{\mathbf{U}}^i)_j = k\,F_j^i\,,$$

hence from (33)

$$(42) \qquad m_{jj}(\mathrm{W}_j^i - \mathrm{W}_j^{i-1}) + k(\mathbf{A}\widehat{\mathbf{W}}^i)_j =$$
$$= k\,\mathrm{F}_j^i - \frac{\lambda}{2(\delta_0+1)}\, m_{jj}((t^* - t^i) - (t^* - t^{i-1})) - \frac{\lambda}{2(\delta_0+1)}\, k(\widehat{\mathbf{A}\Theta^*})_j .$$

Then from (6), (29) and (39) we obtain

$$(43) \qquad m_{jj}(\mathrm{W}_j^i - \mathrm{W}_j^{i-1}) + k(\widehat{\mathbf{A}\mathbf{W}^i})_j \leq -k\,\frac{\lambda}{2}\, m_{jj} < 0 .$$

Then the P.D.M.P. implies that $\{\widehat{\mathbf{W}}^i\}_{i=0}^m$ takes its maximum in D on a node (x_p, t^q) on $\partial_p \mathrm{D}$. Clearly $\mathrm{W}_p^q > 0$, so that $\mathrm{U}_p^q > 0$ and hence $(x_p, t^q) \notin \Gamma_{hk}$. It follows that

$$(44) \qquad \mathrm{d}\,((x_p, t^q), \partial_p \mathrm{Q}_r(x_*, t^*)) < h \qquad \text{or} \qquad < k^{1/2} ,$$

then

$$(45) \qquad \sigma^*(x_p) > (r-h)^2 \geq \frac{r^2}{4} \qquad \text{or} \qquad (t^* - t^q) > r^2 - k \geq \frac{r^2}{4} .$$

On the other hand, $\mathrm{W}_p^q > 0$ implies $\mathrm{U}_p^q > \dfrac{\lambda}{2(\delta_0+1)}(\sigma^*(x_p) + (t^* - t^q))$ hence the assertion (38) is verified with

$$(46) \qquad \gamma_0 = \frac{\lambda}{8(\delta_0+1)} .$$

■

Now we shall bound the distance between the free boundaries by means of the L^∞-error between u and u_{hk}. In order to do that, we will assume that the time-step k is chosen as $k = c\,h^\alpha$ ($\alpha \leq 2$); moreover we assume that an L^∞-error estimate of the type (47) is known (see, e.g., Cortey Dumont [6]; Pietra and Verdi [11]; Fetter [12]):

$$(47) \qquad \sup_i \| u(t^i) - u_{hk}^i \|_{\mathrm{L}^\infty(\Omega)} \leq \eta^2(h)$$

$$\text{with } \lim_{h\to 0} \eta(h) = 0 , \quad \frac{\eta(h)}{h^{\alpha/2}} \geq 2\beta\sqrt{\gamma_0} \qquad \text{for } h \text{ small enough,}$$

were γ_0 is given by (46) and $\beta = \max\left(1, \sqrt{\dfrac{c}{3}}\right)$.

In order to obtain the final Theorem on the rate of convergence of the discrete free boundaries, we state two lemmata.

LEMMA 6. *There exists a positive constant* h_0 *such that, for any* $0 < h \leq h_0$, *for any node* $(x_*, t^*) \in Q$ *and for any* $r > 0$ *with* $Q_r(x_*, t^*) \cap \partial_p^+ Q = \phi$ *and* $\eta^2(h) \leq \gamma_0 r^2$, *we have*

$$u(x, t) \equiv 0 \quad \text{in} \quad Q_r(x_*, t^*) \cap Q => u_{hk}(x_*, t^*) = 0 . \tag{48}$$

Proof. Assume that $u(x, t) \equiv 0$ in $Q_r(x_*, t^*) \cap Q$ and suppose that $u_{hk}(x_*, t^*) > 0$. By (47), $r^2 \geq \dfrac{\eta^2(h)}{\gamma_0} \geq 4\beta^2 h^\alpha \geq \dfrac{4}{3} k$ and $r \geq 2 h^{\alpha/2} \geq \geq 2h$. Then we can apply Theorem 5 and obtain

$$\max_{(x_j, t^i) \in Q_r(x_*, t^*) \cap Q} U_j^i > \gamma_0 r^2 \geq \eta^2(h) . \tag{49}$$

So the error estimate (47) is contradicted.

■

LEMMA 7. Set $\varepsilon_1(h) = \eta(h) \dfrac{(\sqrt{3} + 2)}{2\sqrt{\gamma_0}}$. There exists a positive constant h_0 such that, for any $0 < h \leq h_0$, we have

$$(Q - P) - \mathbf{S}_{\varepsilon_1(h)}(\Gamma) \subset Q - P_{hk} . \tag{50}$$

Proof. Let $\varepsilon_2(h) = \dfrac{\eta(h)}{\sqrt{\gamma_0}}$ and $S = (Q - P) - \mathbf{S}_{\varepsilon_2(h)}(\Gamma)$. If (x_*, t^*) is a node belonging to S, then $u(x, t) \equiv 0$ in $Q_{\varepsilon_2(h)}(x_*, t^*) \cap Q$ and $Q_{\varepsilon_2(h)}(x_*, t^*) \cap \cap \delta_p^+ Q = \phi$. Hence, Lemma 6 yields $u_{hk}(x_*, t^*) = 0$.

The strip $\mathbf{S}_{\varepsilon_2(h)}(\Gamma)$ is not large enough. Indeed, there could exist an element of $\mathcal{T}_{hk}$ with a vertex in S and a vertex in P_{hk}. It is sufficient to remove a further strip of width $\sqrt{3}\,\beta\, h^{\alpha/2}$ from S. Globally we remove a strip of width $\varepsilon_1(h)$ from $Q - P$, and then (50) holds.

■

THEOREM 6. *For any* $\delta > 0$, *there exist two positive constants* h_0 *and* C *such that, for any* $0 < h \leq h_0$, *we have*

$$\mu((P \div P_{hk})^\delta) \leq C\, \eta(h) . \tag{51}$$

Proof. Lemma 7 implies that

$$(P_{hk} - P)^\delta = ((Q - P) - (Q - P_{hk}))^\delta \subset S_{\varepsilon_1(h)}(\Gamma^\delta) . \tag{52}$$

On the other hand, if $(x, t) \in (P - P_{hk})^\delta$, we have $u_{hk}(x, t) = 0$ and then $0 < u(x, t) \leq C_1^2 \eta^2(h)$, by means of the error estimate (47). Hence, we obtain

$$(P \div P_{hk})^\delta \subset \mathbf{S}_{\varepsilon_1(h)}(\Gamma^\delta) \cup A^\delta_{C_1\eta(h)} . \tag{53}$$

Theorem 3 yields $\mu(\mathbf{S}_{\varepsilon_1(h)}(\Gamma^\delta)) \leq C\, \varepsilon_1(h)$ and Theorem 2 yields $\mu(A^\delta_{C_1\eta(h)}) \leq$ $\leq C\, \eta(h)$. Then the thesis follows.

■

Remark 2. If we assume the following property

$$\forall \delta > 0, \quad \exists \varepsilon_0, \quad C > 0: \quad A^\delta_\varepsilon \subset \mathbf{S}_{C\varepsilon}(\Gamma^\delta), \qquad \forall\, 0 < \varepsilon \leq \varepsilon_0, \tag{54}$$

then the distance between the free boundaries can be estimated. Indeed from (53) it follows that

$$(\Gamma_{hk})^\delta \subset \mathbf{S}_{C\eta(h)}(\Gamma^\delta) . \tag{55}$$

Actually, property (54) could be obtained from the non-degeneracy property (7) if the continuous free boundary is regular.

CONCLUDING REMARK. Here, the error estimates (51) and (55) for the free boundaries have been obtained in the cylinder Q. In Pietra and Verdi [11], where the approximation of the free boundary for the multidimensional one-phase Stefan problem has been treated, bounds of the same feature have been derived at each time level. There, the positivity of u_t allowed us to reduce the problem to an elliptic case. In a general parabolic variational inequality, the lack of monotonicity requires the different analysis here used.

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