Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Soluble Groups with Many Černikov Quotients

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **79** (1985), n.1-4, p. 19–24.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1985_8_79_1-4_19_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1985.

Teoria dei gruppi. — *Soluble Groups with Many Černikov Quotients.* Nota (*) di SILVANA FRANCIOSI e FRANCESCO DE GIOVANNI, presentata dal Socio G. ZAPPA.

RIASSUNTO. — Si studiano i gruppi risolubili non di Černikov a quozienti propri di Černikov. Nel caso periodico tali gruppi sono tutti e soli i prodotti semidiretti $H \ltimes N$ con N p–gruppo abeliano elementare infinito e H gruppo irriducibile di automorfismi di N che sia infinito e di Černikov. Nel caso non periodico invece si riconduce tale studio a quello dei moduli a quozienti propri artiniani su un gruppo risolubile finito, e si fornisce una caratterizzazione di tali moduli.

§ 1. INTRODUCTION

If **X** is a class of groups, a group G is said to be *just-non-***X** if it is not in **X** but all its proper quotients are **X**-groups.

Soluble just-non-polycyclic groups are studied in recent papers of Groves [2] and Robinson and Wilson [7], while just-infinite groups are considered in earlier papers of McCarthy [3] and Wilson [8].

Our aim here is to study soluble just-non-Černikov groups. Torsion soluble just-non-Černikov groups are described in a satisfactory way by Theorem A: such groups are precisely the semi-direct products $H \ltimes M$, where M is an infinite abelian group of prime exponent and H is an irreducible infinite Černikov group of automorphisms of M. Theorem B reduces the study of non-torsion soluble just-non-Černikov groups to that of just-non-artinian modules over a finite soluble group, while Theorem C gives a description of such modules.

Finally in § 4 we construct many examples of just-non-Černikov groups, and we embed every Černikov group in a just-non-Černikov group.

Most of our notation is standard. In particular we refer to [5].

§ 2. TORSION SOLUBLE JUST-NON-ČERNIKOV GROUPS

We shall prove:

THEOREM A. *A torsion soluble group* G *is just-non-Černikov if and only if* $G = H \ltimes M$ *where* M *is an infinite abelian group of prime exponent and* H *is*

(*) Pervenuta all'Accademia il 12 settembre 1985.

an infinite Černikov group which acts faithfully as an irreducible group of automorphisms of M.

Proof. Let G be just-non-Černikov; then the last non-trivial term A of the derived series of G is obviously a p-group (for some prime p). Denote by D the divisible part of A and by S the socle of D. If $S \neq 1$, D/S is in Min and so D is in Min, which is impossible. Therefore $D = 1$ and A is reduced. It follows that $A^p = 1$, since otherwise A/A^p is finite and A is finite, a contradiction; thus A is an infinite elementary abelian p-group. For each $a \in A \setminus \{1\}$, the subgroup $C_G(a)$ has infinite index in G, and so we have $B \nleq C_G(a)$ where B/A is the finite residual of G/A. Since A/a^G is finite and B/A is divisible, we obtain

$$[A, B] \leq a^G \text{ , and so } 1 \neq [A, B] \leq \bigcap_{a \in A \setminus \{1\}} a^G = M.$$

Obviously M is the unique minimal normal subgroup of G. Write $Q = G/M$. Then M is a simple Q-module and, if R denotes the finite residual of Q, we have $H^0(R, M) = 0$ since $|Q : R| < \infty$. By a theorem of Robinson ([6] Theorem B) it follows that $H^2(Q, M) = 0$ and so $G = H \ltimes M$ for some $H \leq G$. If $C = C_G(M)$, we have $C = HM \cap C = M(H \cap C)$, and $H \cap C$ is normal in $HM = G$. Since $H \simeq G/M$ is Černikov it follows that $H \cap C = 1$ and $C_G(M) = M$. Therefore H acts faithfully as an irreducible group of automorphisms of M.

Conversely, if $G = H \ltimes M$ has this structure, G is not a Černikov group, since M is infinite. If N is a non-trivial normal subgroup of G, we have $M \cap N \neq 1$, since $C_G(M) = M$, and so $M \leq N$ and G/N is a Černikov group. ∎

Remark. *If* M *is an infinite abelian group of prime exponent p and* H *is an irreducible infinite Černikov group of automorphisms of* M, *then the finite residual* R *of* H *is a p'-group.*

Proof. By contradiction suppose that the p-component K of R is non-trivial, and denote by K_n the subgroup of the elements of order $\leq p^n$ of K; then $K = \bigcup_{n \in N} K_n$ and every K_n is finite. By Theorem A the group $G = H \ltimes M$ is just-non-Černikov and every $L_n = K_n \ltimes M$ is nilpotent (see [5] Part 2, Lemma 6.34). Therefore $1 \neq Z(L_n) \lhd G$, so that $G/Z(L_n)$ is a Černikov group and $L_n/Z(L_n)$ is finite since it has finite exponent. Then L'_n is a finite normal subgroup of G, so that $L'_n = 1$ and each L_n is abelian. Therefore K acts trivially on M and $K = 1$. ∎

In § 4 we will construct a torsion soluble just-non-Černikov group whose unique minimal normal subgroup is not a Hall subgroup.

Example. Let K be the algebraic closure of the field GF (p); then the multiplicative group K^* of K is a direct product of groups of Prüfer type, one for each prime other than p. Let H be an infinite subgroup of K^* satisfying the minimal condition on subgroups, and let A be the additive group of the subfield of K generated by H; then A is an infinite elementary abelian p-group on which H acts faithfully and irreducibly by multiplication. The split extension $G = H \ltimes A$ is just-non-Černikov.

This example is essentially due to Carin. ■

§ 3. NON-TORSION SOLUBLE JUST-NON-ČERNIKOV GROUPS

The analysis of the Fitting subgroup and of the Fitting quotient is essential in describing non-torsion soluble just-non-Černikov groups. In fact we have:

THEOREM B. (1) *Let* G *be a non-torsion soluble just-non-Černikov group and let* F *be the Fitting subgroup of* G. *Then* $Q = G/F$ *is a finite group and* F *is a faithful just-non-artinian* Q-*module.*

(2) *Conversely, if* Q *is a finite soluble group and* F *is a faithful just-non-artinian* Q-*module, every extension of* F *by* Q *is a non-torsion soluble just-non-Černikov group with Fitting subgroup* F.

Proof. (1) Let A be the last non-trivial term of the derived series of G, and let M be the intersection of all non-trivial G-invariant subgroups of A. If $M \neq 1$, M is a minimal normal subgroup of G and $G/C_G(M)$ is an irreducible locally finite group of automorphisms of M, so that M is torsion (see [5] Part 1, Lemma 5.26), which is impossible. Therefore $M = 1$. Denote by B/A the finite residual of the Černikov group G/A. If H is a non-trivial G-invariant subgroup of A, the torsion group $B/C_B(A/H)$ is finite (see [5] Part 1, Theorem 3.29.2) and so $C_B(A/H) = B$ since B/A is divisible; it follows that $[A, B] \leq$ $\leq M = 1$, and so $A \leq Z(B)$ and B is nilpotent. Therefore G is nilpotent-by-finite; it follows that F is a torsion-free nilpotent group and Q is finite.

The group $F/Z(F)$ is Černikov, so that $F' = 1$ and F is abelian. Then $C_G(F) = F$ and F is a faithful Q-module. If K is a non-trivial Q-submodule of F, we have $K \triangleleft G$ and obviously F/K is an artinian Q-module. Finally the Q-module F is non-artinian since G has no minimal normal subgroups.

(2) Let G be an extension of F by Q. Then $C_G(F) = F$ and so, if N is a non-trivial normal subgroup of G, we have that $N \cap F$ is a non-trivial Q-submodule of F and $F/N \cap F$ is an artinian Q-module. Thus $G/N \cap F$ is in Min-n, and so $F/N \cap F$ is in Min (see [5] Part 1, Theorem 5.21); it follows that G/N is Černikov and G is a just-non-Černikov group. Since Q is finite, we obtain that F is torsion-free by Dietzmann's Lemma.

By (1) the Fitting subgroup $F(G)$ of G is abelian, so that $F(G) \leq C_G(F) =$ $= F$ and $F(G) = F$. ■

From Theorem B it follows that it is sufficient to study faithful just-non-artinian modules over finite soluble groups. We recall that a Q-module A is said to have *finite* Q-*rank* r if every finitely generated Q-submodule of A can be generated by $s \leq r$ elements and r is the least positive integer with this property, while A is said to have *finite total* Q-*rank* if the sum of the Q-rank of A/T and of the Q-ranks of the A_p (for all primes p) is finite (where T is the torsion subgroup of A and A_p is the p-component of A). Here $\mathcal{Q}$ denotes the field of rational numbers.

THEOREM C. *Let* Q *be a finite soluble group and let* A *be a faithful* Q-*module. Then* A *is a just-non-artinian* Q-*module if and only if* A *is* **Z**-*torsion-free*, $A \otimes_Z \mathcal{Q}$ *is a simple* $\mathcal{Q}$Q-*module and the* Q-*sections of* A *have finite total* Q-*rank*.

Proof. Let A be just-non-artinian. Since Q is finite, the abelian group A is obviously torsion-free. Let M be a non-trivial $\mathcal{Q}$Q-submodule of $A \otimes_Z \mathcal{Q}$; then $M^* = \{a \in A / a \otimes 1 \in M\}$ is a non-trivial Q-submodule of A and there is an exact sequence

$$M^* \otimes_Z \mathcal{Q} \overset{\alpha}{\rightarrowtail} A \otimes_Z \mathcal{Q} \twoheadrightarrow A/M^* \otimes_Z \mathcal{Q}$$

(see [1] Theorem 60.6). The abelian group A/M^* is in Min, since it is an artinian Q-module (see [5] Part 1, Theorem 5.21), and so

$$(A \otimes_Z \mathcal{Q})/\mathrm{Im}\,\alpha \simeq A/M^* \otimes_Z \mathcal{Q} = 0 \text{ and } M = A \otimes_Z \mathcal{Q} \text{ since } \mathrm{Im}\,\alpha \leq M.$$

If $x \in A \setminus \{0\}$, the Q-submodule xQ of A generated by x is a free abelian group of finite rank and, as above, the abelian group A/xQ is in Min, so that the torsion-free abelian group A has finite (total) rank. It follows that A has finite (total) Q-rank since each finitely generated Q-submodule is finitely generated as a subgroup of A. Let U/V be a Q-section of A with $V \neq 0$. Then A/V is an abelian group in Min and so U/V has finite total rank as an abelian group, and hence also finite total Q-rank.

Conversely let B be a non-trivial Q-submodule of A. We have $0 \neq B \simeq B \otimes_Z \mathbf{Z} \leq B \otimes_Z \mathcal{Q}$ since B is torsion free, and so $B \otimes_Z \mathcal{Q} \neq 0$. There is an exact sequence

$$B \otimes_Z \mathcal{Q} \overset{\alpha}{\rightarrowtail} A \otimes_Z \mathcal{Q} \twoheadrightarrow A/B \otimes_Z \mathcal{Q}$$

and so $A/B \otimes_Z \mathcal{Q} = 0$ since Im α is a non-trivial $\mathcal{Q}$Q-submodule of the simple $\mathcal{Q}$Q-module $A \otimes_Z \mathcal{Q}$. If T/B is the torsion subgroup of A/B, it follows that $A/T \otimes_Z \mathcal{Q} = 0$, and so $A/T = 0$ since it is torsion-free. Therefore A/B is a torsion group and it has finitely many non-trivial primary components since it has finite total Q-rank. Let $H = \langle x_1, \dots, x_n \rangle$ be a finitely generated subgroup of A; since $H^Q = x_1 Q \cdot \ldots \cdot x_n Q$ is a finitely generated Q–submodule of A, there exist $y_1, \dots, y_r$ in A such that $H^Q = y_1 Q \cdot \ldots \cdot y_r Q$ where r is the Q-rank of A. If $|Q| = q$, then H^Q is a free abelian group of rank $\leq rq$. Therefore A has finite rank as an abelian group, and so the abelian group A/B is in Min. Since A is torsion-free, it follows that A is a just-non-artinian Q-module. ■

REMARK 1. *Every soluble non-torsion just-non-Černikov group* G *is a residually finite minimax group.*

Proof. The Fitting subgroup F of G is torsion-free abelian and $|G:F| < \infty$, so that, if $x \in F \setminus \{1\}$, x^G is a free abelian group of finite rank and F/x^G is in Min; then F is a minimax group and it is residually finite (see [4] Lemma 2.21). It follows that G is a residually finite minimax group. ■

REMARK 2. *Let* G *be a group with* $Z(G) \neq 1$. *Then* G *is just-non-Černikov if and only if it is isomorphic with a non-trivial subgroup of* $\mathcal{Q}_\pi$ *(the additive group of all rational numbers whose denominators are* π*-numbers) for some finite set* π *of prime numbers.*

Proof. Suppose G just-non-Černikov. Then $G/Z(G)$ is a Černikov group and so G' is Černikov (see [5], Part 1, Theorem 4.23). Therefore $G' = 1$ and G is a torsion-free abelian group of rank 1, and so it is isomorphic with a subgroup G^* of $\mathcal{Q}(+)$. If $m/n \in G^* \setminus \{0\}$, the set π of the prime numbers which either divide n or *are* orders of elements of $G^*/<m/n>$ is finite, since $G^*/<m/n>$ is in Min. It is easily proved that $G^* \subseteq \mathcal{Q}_\pi$. ■

EXAMPLE. If π is a finite set of prime numbers and α is the automorphism $x \mapsto -x$ of $\mathcal{Q}_\pi$, the group $\langle \alpha \rangle \ltimes \mathcal{Q}_\pi$ is a soluble non-torsion just-non-Černikov group which is non-abelian. ■

§ 4. WREATH PRODUCTS AND JUST-NON-ČERNIKOV GROUPS

In this section we give methods to construct many examples of just-non-Černikov groups.

4.1. *Let* H *be a non-abelian just-non-Černikov group, and let* K *be a finite group. Then* $G = H \wr K$ *is just-non-Černikov.*

Proof. If N is a non-trivial normal subgroup of G, the intersection of N with each component H_i of the base group B is non-trivial, since $Z(H) = 1$. Therefore every $H_i(N \cap B)/N \cap B$ is a Černikov group and so $B/N \cap B$ is Černikov. It follows that G/N is a Černikov group since $|G:B| < \infty$. ■

4.2. *Let* H *be a non-abelian simple group, and let* K *be an infinite Černikov group. Then* $G = H \wr K$ *is just-non-Černikov.*

Proof. Each non-trivial normal subgroup N of G contains the base group of G, and so G/N is Černikov. Moreover G is not a Černikov group since K is infinite. ■

EXAMPLE. Let H be a soluble torsion just-non-Černikov group whose unique minimal normal subgroup has exponent p, and let K be a finite soluble group whose order is divisible by p. Then $G = H \wr K$ is a soluble torsion just-non-Černikov group whose unique minimal normal subgroup is not a Hall subgroup. ■

By 4.2 it follows that every Černikov group is a subgroup and a quotient of a just-non-Černikov group.

4.3. *Let* $G = H \wr K$ *be a just-non-Černikov group with* H *non-simple. Then* H *is just-non-Černikov and* K *is finite.*

Proof. Let B be the base group and denote by L a proper non-trivial normal subgroup of H; then $L \lhd B$ and $L^G = L^K$. Since G/L^G is Černikov, also $B/L^K \simeq \mathrm{Dr}_{k \in K}\, H^k/L^k$ is Černikov, and from $H^k/L^k \neq 1$ it follows that K is finite. We have $H \cap L^K = L$ and so $H/L \simeq HL^K/L^K \leq G/L^K$ is a Černikov group. Finally H is not a Černikov group since G is not Černikov. ■

Our last result is about the cardinality of soluble just-non-Černikov groups

4.4. *A soluble just-non-Černikov group* G *is countable.*

Proof. If G is non-torsion the result follows from Remark 1. Suppose that G is a torsion group and so $G = H \ltimes M$ as in Theorem A. Then H is countable since it is Černikov and M is countable because it is an irreducible (and so cyclic) $\mathbf{Z}_p H$-module. ■

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