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Analisi matematica. — *Characterization of some interpolation spaces* (I). Nota di ALESSANDRA LUNARDI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si calcolano alcuni spazi di interpolazione fra spazi di funzioni hölderiane.

1. DEFINITIONS AND PRELIMINARIES

X and Y will denote two Banach spaces, with Y continuously imbedded in X (we shall write $Y \hookrightarrow X$).

DEFINITION 1.1. For every $\theta \in]0,1[$ set:

$$W(\theta ; Y, X) = \{u :]0,1] \to X ; t \to t^\theta u(t) \in L^\infty([0,1] ; Y) ; \quad t \to t^\theta u'(t) \in L^\infty([0,1] ; X)\}$$

$$C(\theta ; Y, X) = \{u :]0,1] \to X ; t \to t^\theta u(t) \in C([0,1] ; Y), \quad t \to t^\theta u'(t) \in C([0,1] ; X) ; \lim_{t\to 0^+} \| t^\theta u(t) \|_Y = = \lim_{t\to 0^+} \| t^\theta u'(t) \|_X = 0\}.$$

$W(\theta ; Y, X)$ and $C(\theta ; Y, X)$ are Banach spaces under the norm:

$$\| u \|_{(\theta ; Y, X)} = \| t^\theta u(t) \|_{L^\infty([0,1], Y)} + \| t^\theta u'(t) \|_{L^\infty([0,1]; X)}.$$

One can show that if u belongs to $W(\theta ; Y, X)$ then there exists $X - \lim_{t\to 0^+} u(t)$. Then we make the following definition:

DEFINITION 1.2. For every $\theta \in]0,1[$ set:

$$(Y, X)_{\theta,\infty} = \{u(0) ; u \in W(\theta ; Y, X)\}$$

$$(Y, X)_\theta = \{u(0) ; u \in C(\theta ; Y, X)\}$$

$(Y, X)_{\theta,\infty}$ and $(Y, X)_\theta$ are Banach spaces under the respective norms:

$$\| a \|_{\theta,\infty} = \inf_{\substack{u(0)=a \\ u \in W(\theta; Y, X)}} \| u \|_{(\theta; Y, X)} \qquad \forall a \in (Y, X)_{\theta,\infty} \tag{1.1}$$

$$\| a \|_\theta = \inf_{\substack{u(0)=a \\ u \in C(\theta; Y, X)}} \| u \|_{(\theta; Y, X)} \qquad \forall a \in (Y, X)_\theta . \tag{1.2}$$

(*) Nella seduta del 9 gennaio 1982.

Observe that $(Y, X)_{\theta,\infty}$ is the space $S(\infty, 1-\theta, X, -\theta, Y)$ of Lions-Peetre (see [3] pp. 39-43). Clearly we have: $Y \hookrightarrow (Y, X)_\theta \rightarrow (Y, X)_{\theta,\infty}$. Moreover one can show that Y is dense in $(Y, X)_\theta$ (while Y is generally not dense in $(Y, X)_{\theta,\infty}$).

DEFINITION 1.3. Let $A: D(A) \subset X \to X$ be a linear operator, infinitesimal generator of a bounded semigroup e^{tA} in X. For every $\theta \in]0,1[$ and $k \in \mathbf{N}$ set:

$$D_A(\theta, \infty) = (D(A), X)_{1-\theta,\infty} \; ; \; D_A(\theta) = (D(A), X)_{1-\theta}$$

$$D_A(\theta + k, \infty) = \{x \in D(A^k) \; ; \; A^k x \in D_A(\theta, \infty)\}$$

$$D_A(\theta + k) = \{x \in D(A^k) \; ; \; A^k x \in D_A(\theta)\}$$

and let:

$$\| x \|_{D_A(\theta,\infty)} = \| x \|_{(D(A),X)_{1-\theta,\infty}} \qquad \forall x \in D_A(\theta, \infty)$$

$$\| x \|_{D_A(\theta)} = \| x \|_{(D(A),X)_{1-\theta}} \qquad \forall x \in D_A(\theta)$$

$$\| x \|_{D_A(\theta+k,\infty)} = \| x \|_x + \| A^k x \|_{D_A(\theta,\infty)} \qquad \forall x \in D_A(\theta + k, \infty)$$

$$\| x \|_{D_A(\theta+k)} = \| x \|_x + \| A^k x \|_{D_A(\theta)} \qquad \forall x \in D_A(\theta + k) .$$

PROPOSITION 1.4. *For every* $\theta \in]0, 1[$ *we have*:

$$D_A(\theta, \infty) = \{x \in X \; ; \; \sup_{t \in]0,1]} \| t^{-\theta} (e^{tA} x - x) \|_X < \infty\}$$

$$D_A(\theta) = \{x \in X \; ; \; \lim_{t \to 0^+} \| t^{-\theta} (e^{tA} x - x) \|_X = 0\}$$

and the norm:

$$\| x \| = \| x \|_X + \| t^{\theta} (e^{tA} x - x) \|_{L^\infty([0,1];X)}$$

is equivalent to the norm of $D_A(\theta, \infty)$ *and to the norm of* $D_A(\theta)$. The proof can be found in Grisvard [2] p. 667 for the case $D_A(\theta, \infty)$ and in Da Prato-Grisvard [1] p. 336 for the case $D_A(\theta)$.

Let now $A_i: D(A_i) \subset X \to X$, $i = 1, \cdots, n$ be linear operators, satisfying the assumptions of Definition 1.3. Suppose that:

$$R(t, A_i) R(t, A_j) = R(t, A_j) R(t, A_i) \qquad \forall t > 0 \quad \forall i, j = 1, \cdots, n .$$

For every $m \in \mathbf{N}$ set:

$$K^m = \bigcap_{\substack{k_1, \cdots, k_n \in \mathbf{N} \\ k_1 + \cdots + k_n = m}} D(A_1^{k_1} \cdot \ldots \cdot A_n^{k_n})$$

and let:

$$\| x \|_{K^m} = \| x \|_X + \sum_{\substack{k_1, \cdots, k_n \in \mathbf{N} \\ k_1 + \cdots + k_n = m}} \| A_1^{k_1} \cdot \ldots \cdot A_n^{k_n} x \|_X .$$

PROPOSITION 1.5. *Let* $m \in \mathbf{N}$, $\theta \in]0,1[$ *be such that* $\theta m \notin \mathbf{N}$.

Let $k = [\theta m]$ and $\sigma = \theta m - k$. Set:

$$B_1(t)\,x = \sum_{i=1}^{n} t^{-\sigma} \| (e^{tA_i} - 1) A_i^k x \|_X \qquad \forall t \in]0,1] \; ; \; \forall x \in \bigcap_{i=1}^{n} D(A_i^k)$$

$$B_2(y)\,x = \sum_{\substack{k_1,\dots,k_n \in \mathbf{N} \\ k_1+\dots+k_n=k}} t^{-\sigma} \| (e^{y_1 A_1} - 1) \cdot \ldots \cdot (e^{y_n A_n} - 1) A_1^{k_1} \cdot \ldots \cdot A_n^{k_n} x \|_X$$

$$\forall y \in (]0,1])^n \; ; \; \forall x \in K^k .$$

Then we have:

$$(1.4) \qquad (K^m, X)_{1-\theta,\infty} = \left\{ x \in \bigcap_{i=1}^{n} D(A_i^k) \; ; \; \sup_{t \in]0,1]} B_1(t)\,x < \infty \right\} = \{ x \in K^k \; ; \; \sup_{y \in (]0,1])^n} B_2(y)\,x < \infty \}$$

$$(1.5) \qquad (K^m, X)_{1-\theta} = \left\{ x \in \bigcap_{i=1}^{n} D(A_i^k) \; ; \; \lim_{t \to 0^+} \|B_1(t)\,x\|_X = 0 \right\} = \{ x \in K^k \; ; \; \lim_{y \to 0} \| B_2(y)\,x \|_X = 0 \}$$

and the following norms are both equivalent to those of $(K^m, X)_{1-\theta,\infty}$ *amd* $(K^m, X)_{1-\theta}$:

$$(1.6) \qquad \| x \|^{(1)} = \| x \|_X + \sup_{t \in]0,1]} \| B_1(t)\,x \|_X$$

$$(1.7) \qquad \| x \|^{(2)} = \| x \|_X + \sup_{y \in (]0,1])^n} \| B_2(y)\,x \|_X .$$

The proof, in the case $(K^m, X)_{1-\theta,\infty}$ can be found in Triebel [4] pag. 88; in the case $(K^m, X)_{1-\theta}$ it follows from the density of K^m in $(K^m, X)_{1-\theta}$.

Remark 1.6. From (1–4) and (1–6) it follows that

$$(K^m, X)_{1-\theta,\infty} = \bigcap_{i=1}^{n} D_{A_i}(m\,\theta \; ; \; \infty)$$

$$(K^m, X)_{1-\theta} = \bigcap_{i=1}^{n} D_{A_i}(m\theta)$$

with equivalence of the respective norms.

2. EVALUATION OF SOME INTERPOLATION SPACES

DEFINITION 2.1. Let Ω be open in $\mathbf{R}^n$. For every $k \in \mathbf{N}$ let $UC^k(\overline{\Omega})$ be the space of all $f \in C^k(\overline{\Omega})$ such that $D^\alpha f$ is uniformly continuous and bounded for every multi-index α with $|\alpha| = k$.

For every $f \in UC^k(\overline{\Omega})$ set:

$$\|f\|_k = \sup_{x \in \overline{\Omega}} |f(x)| + \sum_{|\alpha|=k} \sup_{x \in \overline{\Omega}} |D^\alpha f(x)| .$$

For every $k \in \mathbf{N}$, $\sigma \in]0,1[$ let $C^{k,\sigma}(\overline{\Omega})$ be the space of all $f \in UC^k(\overline{\Omega})$ such that $D^\alpha f$ is Hölder—continuous with exponent σ for every multi—index α with $|\alpha| = k$. For every $f \in C^{k,\sigma}(\overline{\Omega})$ set:

$$\|f\|_{k,\sigma} = \|f\|_k + \sum_{|\alpha|=k} \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\sigma} .$$

Finally let $h^{k,\sigma}(\overline{\Omega})$ be the subspace of $C^{k,\sigma}(\overline{\Omega})$ consisting of all f such that

$$\lim_{\tau \to 0} \sup_{\substack{x,y \in \overline{\Omega} \\ |x-y| \leq \tau}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\tau^\sigma} = 0 \qquad \forall \alpha \in \mathbf{N}^n, |\alpha| = k$$

$h^{k,\sigma}(\overline{\Omega})$ is a Banach space under the $C^{k,\sigma}$-norm.

PROPOSITION 2.2. *Let* $m \in \mathbf{N}$, $\theta \in]0,1[$ *be such that* $m\theta \notin \mathbf{N}$; *let* $k = [m\theta]$ *and* $\sigma = m\theta - k$. *Then we have*:

$$(UC^m(\mathbf{R}^n), UC^0(\mathbf{R}^n))_{1-\theta,\infty} = C^{k,\sigma}(\mathbf{R}^n)$$
$$(UC^m(\mathbf{R}^n), UC^0(\mathbf{R}^n))_{1-\theta} = h^{k,\sigma}(\mathbf{R}^n)$$

with equivalence of the respective norms.

Proof. For every $i = 1, \cdots, n$ let $D(A_i) = \Big\{ f \in UC^0(\mathbf{R}^n) ; \forall x \in \mathbf{R}^n$ $f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n) \in C^1(\mathbf{R}) ; \frac{\partial f}{\partial x_i} \in UC^0(\mathbf{R}^n) \Big\}$; $A_i f(x) = \frac{\partial f}{\partial x_i}(x)$ $\forall x \in \mathbf{R}^n$, $\forall f \in D(A_i)$. Then A_i is the infinitesimal generator of the semigroup $e^{tA_i} f = f(x + te_i)$ in $UC^0(\mathbf{R}^n)$. Using the notations of Proposition 1.5 we have: $K^m = UC^m(\mathbf{R}^n)$; the statement follows now from (1.5) and (1.7) of Proposition 1.5.

COROLLARY 2.3. *Let* $\alpha, \theta \in]0,1[$, $k \in \mathbf{N}$ *be such that* $\alpha + k\theta \notin \mathbf{N}$. *Let* $m = [\alpha + k\theta]$ *and* $\sigma = \alpha + k\theta - m$. *Then we have*:

$$(C^{k,\alpha}(\mathbf{R}^n), C^{0,\alpha}(\mathbf{R}^n))_{1-\theta,\infty} = C^{m,\sigma}(\mathbf{R}^n) \tag{2.1}$$

$$(h^{k,\alpha}(\mathbf{R}^n), h^{0,\alpha}(\mathbf{R}^n))_{1-\theta} = h^{m,\sigma}(\mathbf{R}^n) . \tag{2.2}$$

Proof. It is sufficient to apply Proposition 2.1 and Reiteration Theorem (for (2.2) see Da Prato–Grisvard [1], for (2.1) see Triebel [4], p. 62).

PROPOSITION 2.4. *Let* $\mathbf{R}^n_+$ *be the half-space* $\{x \in \mathbf{R}^n ; x_n \geq 0\}$. *For* $\alpha, \theta \in]0,1[$ *and* $k \in \mathbf{N}$ *such that* $\alpha + k\theta \notin \mathbf{N}$ *let* $m = [\alpha + k\theta]$, $\sigma = \alpha + k\theta - m$.

Then we have:

$$(C^{k,\alpha}(\mathbf{R}^n_+), C^{0,\alpha}(\mathbf{R}^n_+))_{1-\theta,\infty} = C^{m,\sigma}(\mathbf{R}^n_+)$$
$$(h^{k,\alpha}(\mathbf{R}^n_+), h^{0,\alpha}(\mathbf{R}^n_+))_{1-\theta} = h^{m,\sigma}(\mathbf{R}^n_+).$$

Proof. Let $\phi \in C^\infty(\mathbf{R})$ be such that $\phi(t) = 0 \ \forall t \leq -1$ and $\phi(t) = 1 \ \forall t \geq -\frac{1}{2}$. Set, for every $x \in \mathbf{R}^n$:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x_n \geq 0 \\ f(x_1, \cdots, x_{n-1}, 0)\cdot\phi(x_n) & \text{if } x_n \leq 0. \end{cases}$$

Then $f \in (C^{k,\alpha}(\mathbf{R}^n_+), C^{0,\alpha}(\mathbf{R}^n_+))_{1-\theta,\infty} \iff \tilde{f} \in (C^{k,\alpha}(\mathbf{R}^n), C^{0,\alpha}(\mathbf{R}^n))_{1-\theta,\infty}$; $f \in (h^{k,\alpha}(\mathbf{R}^n_+); h^{0,\alpha}(\mathbf{R}^n_+))_{1-\theta} \iff \tilde{f} \in (h^{k,\alpha}(\mathbf{R}^n), h^{0,\alpha}(\mathbf{R}^n))_{1-\theta}$ and $f \in C^{m,\alpha}(\mathbf{R}^n_+) \iff \tilde{f} \in C^{m,\alpha}(\mathbf{R}^n)$, $f \in h^{m,\alpha}(\mathbf{R}^n_+) \iff \tilde{f} \in h^{m,\alpha}(\mathbf{R}^n)$.

The statement follows now from Corollary 2.3.

DEFINITION 2.5. Let $k \in \mathbf{N}$, $\alpha \in]0,1[$ and let $A_1, \cdots, A_m$ be linear differential operators of order $\leq k$ in $\overline{\Omega}$, where $\overline{\Omega}$ is an open set in $\mathbf{R}^n$. Set:

$$C^{k,\alpha}_{A_1,\cdots,A_m}(\overline{\Omega}) = \{f \in C^{k,\alpha}(\overline{\Omega})\ ;\ A_i f(x) = 0 \quad \forall x \in \partial\Omega, i = 1, \cdots, m\}$$
$$h^{k,\alpha}_{A_1,\cdots,A_m}(\overline{\Omega}) = \{f \in h^{k,\alpha}(\overline{\Omega})\ ;\ A_i f(x) = 0 \quad \forall x \in \partial\Omega, i = 1, \cdots, m\}.$$

PROPOSITION 2.6. *Let* $\alpha, \theta \in]0,1[$, $k \in \mathbf{N}$ *such that* $\alpha + k\theta \notin \mathbf{N}$ *and let* I *be the identity map in* $\mathbf{R}^n_+$. *Then, if* $m = [\alpha + k\theta]$, $\sigma = \alpha + k\theta - m$ *we have*:

$$(C^{k,\alpha}_I(\mathbf{R}^n_+), C^{0,\alpha}_I(\mathbf{R}^n_+))_{1-\theta,\infty} = C^{m,\sigma}_I(\mathbf{R}^n_+)$$
$$(h^{k,\alpha}_I(\mathbf{R}^n_+), h^{0,\alpha}_I(\mathbf{R}^n_+))_{1-\theta} = h^{m,\sigma}_I(\mathbf{R}^n_+).$$

Proof. The statement is a consequence of Proposition 2.4.

PROPOSITION 2.7. *Let* $\alpha, \theta \in]0,1[$, $k \in \mathbf{N}$ *be such that* $\alpha + k\theta \notin \mathbf{N}$.
The we have:

$$(C^{k,\alpha}_{I,(\partial/\partial x_n),(\partial^2/\partial x_n^2),\cdots,(\partial^k/\partial x_n^k)}(\mathbf{R}^n_+), C^{0,\alpha}_I(\mathbf{R}^n_+))_{1-\theta,\infty} = C^{m,\sigma}_{I,(\partial/\partial x_n),\cdots,(\partial^m/\partial x_n^m)}(\mathbf{R}^n_+)$$

$$(h^{k,\alpha}_{I,(\partial/\partial x_n),\cdots,(\partial^k/\partial x_n^k)}(\mathbf{R}^n_+), h^{0,\alpha}_I(\mathbf{R}^n_+))_{1-\theta} = h^{m,\sigma}_{I,(\partial/\partial x_n),\cdots,(\partial^m/\partial x_n^m)}(\mathbf{R}^n_+)$$

where $m = [\alpha + k\theta]$, $\sigma = \alpha + k\theta - m$.

The proof is similar to that of Proposition 2.1.

COROLLARY 2.8. *Let* $\alpha, \theta \in]0,1[$, $k \in \mathbf{N}$ *be such that* $\alpha + k\theta < 1$. *Let* Λ *be a differential operator in* $\mathbf{R}^n_+$ *of order* $\leq k$. *Then we have*:

$$(C^{k,\alpha}_{I,\Lambda}(\mathbf{R}^n_+), C^{0,\alpha}_I(\mathbf{R}^n_+))_{1-\theta,\infty} = C^{0,\alpha+k\theta}_I(\mathbf{R}^n_+) \tag{2.3}$$

$$(h^{k,\alpha}_{I,\Lambda}(\mathbf{R}^n_+), h^{0,\alpha}_I(\mathbf{R}^n_+))_{1-\theta} = h^{0,\alpha+k\theta}_I(\mathbf{R}^n_+). \tag{2.4}$$

Proof. As $C^{k,\alpha}_{I,(\partial/\partial x_n),\ldots,(\partial^k/\partial x_n^k)}(\mathbf{R}^n_+) \hookrightarrow C^{k,\alpha}_{I,\Lambda}(\mathbf{R}^n_+) \hookrightarrow C^{k,\alpha}_{I}(\mathbf{R}^n_+)$, we have: $(C^{k,\alpha}_{I,(\partial/\partial x_n),\ldots,(\partial^k/\partial x_n^k)}(\mathbf{R}^n_+)\,,\ C^{0,\alpha}_{I}(\mathbf{R}^n))_{1-\theta,\infty} \hookrightarrow (C^{k,\alpha}_{I,\Lambda}(\mathbf{R}^n_+)\,,\ C^{0,\alpha}_{I}(\mathbf{R}^n_+))_{1-\theta,\infty} \hookrightarrow$ $\hookrightarrow (C^{k,\alpha}_{I}(\mathbf{R}^n_+)\,,\ C^{0,\alpha}_{I}(\mathbf{R}^n_+))_{1-\theta}$ and then (2.3) follows from Propositions 2.6 and 2.7. The proof of (2.4.) is analogous.

Let now Ω be an open bounded set in $\mathbf{R}^n$ with $\partial\Omega$ sufficiently regular; for every $x \in \partial\Omega$ let $n(x)$ be the unitary exterior normal vector.

PROPOSITION 2.9. *Let* $\alpha, \theta \in]0,1[$, $k \in \mathbf{N}$ *be such that* $\alpha + k\theta \notin \mathbf{N}$. *Let* $m = [\alpha + k\theta]$, $\sigma = \alpha + k\theta - m$. *Then, if* $\partial\Omega$ *is of class* C^{k+1}, *we have*:

$$(C^{k,\alpha}_{I}(\overline{\Omega})\,,\ C^{0,\alpha}_{I}(\overline{\Omega}))_{1-\theta,\infty} = C^{m,\sigma}_{I}(\overline{\Omega})$$

$$(C^{k,\alpha}_{I,(\partial/\partial n),\ldots,(\partial^k/\partial n^k)}(\overline{\Omega})\,,\ C^{0,\alpha}_{I}(\overline{\Omega}))_{1-\theta,\infty} = C^{m,\sigma}_{I,(\partial/\partial n),\ldots,(\partial^m/\partial n^m)}(\overline{\Omega})$$

$$(h^{k,\alpha}_{I}(\overline{\Omega})\,,\ h^{0,\alpha}_{I}(\overline{\Omega}))_{1-\theta} = h^{m,\sigma}_{I}(\overline{\Omega})$$

$$(h^{k,\alpha}_{I,(\partial/\partial n),\ldots,(\partial^k/\partial n^k)}(\overline{\Omega})\,,\ h^{0,\alpha}_{I}(\overline{\Omega}))_{1-\theta} = h^{m,\sigma}_{I,(\partial/\partial n),\ldots,(\partial^m/\partial n^m)}(\overline{\Omega})\,.$$

Proof. Using the fact that $\overline{\Omega}$ is locally diffeomorphic to $\mathbf{R}^n$ or to $\mathbf{R}^n_+$, we can get the statement by the analogous properties which hold if $\overline{\Omega}$ is replaced by $\mathbf{R}^n$ or $\mathbf{R}^n_+$ (Propositions 2.3, 2.6, 2.7).

COROLLARY 2.10. *Let* $\alpha, \theta \in]0,1[$, $k \in \mathbf{N}$ *be such that* $\alpha + k\theta < 1$. *Let* Λ *be a differential operator of order* $\leq k$ *in* $\overline{\Omega}$. *Then we have*:

$$(C^{k,\alpha}_{I,\Lambda}(\overline{\Omega})\,,\ C^{0,\alpha}_{I}(\overline{\Omega}))_{1-\theta,\infty} = C^{0,\alpha+k\theta}_{I}(\overline{\Omega})$$

$$(h^{k,\alpha}_{I,\Lambda}(\overline{\Omega})\,,\ h^{0,\alpha}_{I}(\overline{\Omega}))_{1-\theta} = h^{0,\alpha+k\theta}_{I}(\overline{\Omega})\,.$$

The proof is quite similar to that of Corollary 2.8.

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