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# Rendiconti

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## A first order partial differential equation with an integral boundary condition

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# RENDICONTI

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## SEZIONE I

**(Matematica, meccanica, astronomia, geodesia e geofisica)**

**Analisi matematica.** — *A first order partial differential equation with an integral boundary condition* (*). Nota (**) di GABRIELLA DI BLASIO (***), MIMMO IANNELLI (****) e EUGENIO SINESTRARI (****), presentata dal Corrisp. E. VESENTINI.


RIASSUNTO. — In questo lavoro si considera un'equazione alle derivate parziali del primo ordine con una condizione sulla frontiera di tipo integrale. Si studia l'esistenza, l'unicità e il comportamento asintotico delle soluzioni.


### 1. INTRODUCTION AND STATEMENT OF THE RESULTS

This paper is concerned with the problem of finding $u(a, t)$ for $a \in [0, A]$ and $t \geq 0$ such that

$$(1) \quad \begin{cases} u_t(a, t) + u_a(a, t) = -m(a)\, u(a, t) & 0 \leq a < A, t \geq 0 \\ u(0, t) = f(E(t))\, E(t) & t > 0 \\ u(a, 0) = u_0(a) & 0 \leq a \leq A \end{cases}$$

where

$$E(t) = \int_0^A b(a)\, u(a, t)\, da .$$

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System (1) can be considered as a nonlinear version of the classical Lotka-Von Foerster model for the evolution in time of an age structured population. In this case the function $u(a, t)$ denotes the population density at time $t$ with respect to age $a$. Moreover the functions $m$ and $b$ denote the age-specific mortality rate and fertility rate, respectively.

We shall study problem (1) under the following assumptions:

($H_1$) $m \geq 0$ is continuous on $[0, A[$ and we have $\int_0^A m(a)\,da = +\infty$

($H_2$) $b \geq 0$ is continuous on $[0, A]$ and $b \not\equiv 0$

($H_3$) $f \geq 0$ is continuous and bounded on $R_+ = [0, +\infty[$ and $F(x) = xf(x)$ is piecewise continuously differentiable on $R_+$

($H_4$) $u_0 \geq 0$ is piecewise continuous on $[0, A]$

($H_5$) $u_0(a) > 0$ for some $a < A'$, where $A' = \sup\{a \in [0, A[, b(a) > 0\}$.

The integral condition in ($H_1$) expresses the fact that A is the maximum attainable age. In fact the surviving probability

$$p(a) = \exp\left(-\int_0^a m(a')\,da'\right)$$

goes to zero as $a \to A$. Moreover by virtue of condition ($H_5$) we avoid trivial situations, as we shall see later.

Our main interest here is to study the stability of the stationary solutions of (1). It can be seen that a stationary solution is given by

$$u(a) = Bp(a)$$

where $B \geq 0$ verifies

$$BRf(BR) = B \tag{2}$$

and R is a constant defined as

$$R = \int_0^A b(a)\,p(a)\,da\,.$$

Concerning the stationary solutions of (1) we shall assume

($H_6$) There exist $B_1$ and $B_2$ with $0 \leq B_1 < B_2$ solutions of equation (2) and we have $f(x) > R^{-1}$ for $x \in ]RB_1, RB_2[$ and $f(x) < R^{-1}$ for $x \notin [RB_1, RB_2]$.

From condition ($H_6$) we find that there exist the trivial solution and the two stationary solutions $u_1(a) = B_1 p(a)$ and $u_2(a) = B_2 p(a)$. If $B_1 = 0$ then $u_2$ is the unique non trivial stationary solution.

We shall study the stability of the equilibria under the following condition for the function $F(x) = xf(x)$

($H_7$) F is non decreasing on $R_+$ and we have $F(x) > 0$ for $x > 0$.

For a discussion of the biological meaning of conditions ($H_6$) and ($H_7$) we refer to Clark [1] and May *et al.* [5].

Under the above assumptions it can be proved that there exists a unique global non negative solution of (1). Moreover, concerning the asymptotic behavior we have the following results: if $B_1 \neq 0$ then the trivial solution and $u_2$ are locally asymptotically stable, whereas $u_1$ is unstable; on the other hand if $B_1 = 0$ then the solutions of (1) tend to the unique non trivial stationary solution $u_2$. The details of the proofs of our results are given in [2].

## 2. Existence and properties of the solutions of (1)

Let now consider the problem of finding solutions of system (1). First we note that, integrating (1) along its characteristics, it can be seen that a function $u$ satisfies (1) if and only if it is given by

$$u(a, t) = \begin{cases} B(t-a)\,p(a) & , \quad \text{if} \quad a < t \\ \dfrac{u_0(a-t)}{p(a-t)}\,p(a) & , \quad \text{if} \quad a \geq t \end{cases} \tag{3}$$

where the function B satisfies the following equation

$$B(t) = F\left(\int_0^t K(t-a)\,B(a)\,da + g(t)\right) \tag{4}$$

with K and $g$ defined as

$$K(a) = \begin{cases} b(a)\,p(a) & , \quad \text{if} \quad a < A \\ 0 & , \quad \text{if} \quad a \geq A \end{cases}$$

and

$$g(t) = \begin{cases} \displaystyle\int_t^A b(a)\,p(a)\,\frac{u_0(a-t)}{p(a-t)}\,da & , \quad \text{if} \quad t < A \\ 0 & , \quad \text{if} \quad t \geq A\,. \end{cases}$$

Therefore in order to investigate the existence and the properties of the solutions of (1) we shall study equation (4).

Concerning existence we have the following result.

THEOREM 1. *Let* $(H_1)$–$(H_4)$ *hold. Then there exists a unique continuous* $B : R_+ \to R_+$, *solution of equation* (4). *Moreover* B *depends continuously on the initial datum* $u_0$.

We now investigate the behavior of the function $B(t)$. As we mentioned in section 1 it can be seen that if $(H_5)$ does not hold then $B \equiv 0$. Hence using (3) we find that the solution of (1) is identically zero for $t \geq A$. Now let us note that if $(H_5)$ holds then there exist $a_i \in ]0, A[$, $(i = 1, 2, 3, 4)$, with $a_1 < a_4$ such that

$$u_0 > 0 \quad \text{on} \quad ]a_1, a_2[\ ; b > 0 \quad \text{on} \quad ]a_3, a_4[. \tag{5}$$

As a consequence of (5) we have the following preliminary result concerning the behavior of $B(t)$.

THEOREM 2. *Let* $(H_1)$–$(H_5)$ *hold and let* $F(x) > 0$ *for* $x > 0$. *Furthermore let* $a_i$ *be given by* (5) *and set* $(a_3 - a_2)_+ = max(0, a_3 - a_2)$. *If* B *satisfies* (4) *and* $n_0$ *is an integer such that*

$$n_0 > \frac{(a_3 - a_2)_+ + a_1 + a_3 - a_4}{a_4 - a_3}$$

*then, setting* $T = (a_3 - a_2)_+ + n_0 a_3$, *we have* $B(t) > 0$ *for* $t \geq T$.

## 3. Asymptotic behavior of the solutions

We now turn to the problem of the stability of the equilibria of (1). To this end we shall study the asymptotic behavior of the solutions of (4).

In what follows we shall assume that $(H_1)$–$(H_7)$ hold. We have:

THEOREM 3. *Let* $A'$ *be defined as in* $(H_5)$ *and let* I *be a closed interval of length* $A'$. *Then the following properties hold*

(i) if $B(t) > B_1$ on I, then $\lim_{t \to \infty} B(t) = B_2$

(ii) if $B(t) < B_1$ on I, then $\lim_{t \to \infty} B(t) = 0$.

Using the continuous dependence of B upon $u_0$ and Theorem 4 we can obtain the following result concerning the stability of the stationary solutions of (1).

THEOREM 4. *Let* $0 < B_1 < B_2$. *Then* (1) *has two non trivial stationary solutions* $u_1(a) = B_1 p(a)$ *and* $u_2(a) = B_2 p(a)$. *Moreover we have*

(i) $u_2$ is locally asymptotically stable

(ii) $u_1$ is unstable

(iii) the trivial solution is locally asymptotically stable.

Finally using Theorems 2 and 3 we can prove the following:

THEOREM 5. *Let* $B_1 = 0$. *Then* (1) *has a unique non trivial stationary solution* $u_2(a) = B_2\, p(a)$, *which is globally asymptotically stable.*

## 4. EXAMPLES

With suitable choices of the function $f$ we can obtain some examples which are relevant in fish population dynamics. We will list only three cases ($c$, $\mathrm{d} > 0$ and $0 < r \leq 1$ are constants):

1) the Beverton-Holt model (see [5]), where

$$f(x) = \frac{c}{\mathrm{d} + x^r}$$

2) the Chapman model (see [5]) where

$$f(x) = \frac{c\,(1 - \exp(-\mathrm{d}x))}{x}$$

3) the depensatory model (see Hoppensteadt ∊[3]) where

$$f(x) = \frac{cx}{\mathrm{d} + x^2}\,.$$

With suitable values of $c$ and $d$ in the first two cases we have that all solutions of (1) tend to the unique non trivial stationary solution of (1). In the latter case we have two non trivial stationary solutions and the behavior of the solutions of (1) is described by Theorem 4.

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