Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Alessandra Lunardi

Analyticity of the maximal solution of an abstract parabolic equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **70** (1981), n.2, p. 75–80.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1981_8_70_2_75_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1981.

Analisi matematica. — *Analyticity of the maximal solution of an abstract parabolic equation.* Nota di ALESSANDRA LUNARDI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Studia l'analiticità della soluzione massimale di una equazione parabolica astratta in spazi di Banach.

INTRODUCTION

Let E be a real Banach space, and F a densely and continuously imbedded subspace in E; moreover let $f : F \to E$ be a continuously differentiable mapping.

We shall consider the following Cauchy problem:

$$(\mathrm{P}) \begin{cases} u'(t) = f(u(t)) \\ u(0) = u_0 . \end{cases}$$

In [2] existence and uniqueness are established for the maximal classical solution $u(t, u_0)$, of (P), which is defined and continuous in a relatively open set $\Omega \subset \mathbf{R}^+ \times F$.

Roughly speaking, the assumptions required in [2] are the following: $f'(u_0)$ is the infinitesimal generator of an analytic semi-group in E, depending continuously on u_0, and F and E are continuous interpolation spaces (see [2]).

In this paper we give a new and simpler proof of this result; the technique applied here allows us to study the analyticity of the solution u with respect to (t, u_0). More precisely, we show that if $f : F_{\mathbf{C}} \to E_{\mathbf{C}}$ (1) is holomorphic (2), then the maximal solution of (P) is analytic in (t, u_0) for $t > 0$. □

DEFINITION 1.

Let E, F be two real (resp. complex) Banach spaces, F being densely and continuously imbedded in E. For every $T > 0$ and $\theta \in]0, \pi]$ set $S_\theta = \{z \in \mathbf{C} ; z \neq 0 , |\arg z| < \theta\}$ and $S_{\theta,T} = S_\theta \cap \{z \in \mathbf{C} ; |z| < T\}$.

(*) Nella seduta del 14 febbraio 1981.

(1) $F_{\mathbf{C}}$, $E_{\mathbf{C}}$ denote respectively the complexifications of F and E.

(2) For definitions and properties of holomorphic functions in Banach spaces, see [1] and [4].

A linear operator $A : F \to E$ is said to satisfy the hypothesis $\mathscr{R} = \mathscr{R}(\theta, M, \omega, c', c'', \gamma)$ if:

(a) $A \in L(F, E)$ (3); there exist $c', c'' > 0$ such that

$$c'(\|y\|_E + \|Ay\|_F) \leq \|y\|_F \leq c''(\|y\|_E + \|Ay\|_F) \qquad \forall y \in F$$

(b) A is the infinitesimal generator of an analytic semi-group $e^{tA} : R^+ \to L(E)$ (3) $\left(\text{resp. } e^{\lambda A} : S_\theta \to L(E) \text{ for some } \theta \in \left]0, \frac{\pi}{2}\right]\right)$ and moreover we have:

$\|e^{tA}\|_{L(E)} \leq Me^{\omega t}$ $\forall t > 0$ (resp. $\|e^{\lambda A}\|_{L(E)} \leq Me^{\omega \mathrm{Re}\lambda}$ $\forall \lambda \in \bar{S}_\theta$)

with $M > 0$ and $\omega \in \mathbf{R}$.

(c) There exists a continuous function γ such that for any $\varphi \in C([0, T]; E)$ (4) we have:

$e^{tA} * \varphi$ (6) $\in C([0, T]; F)$ and

$\|e^{tA} * \varphi\|_{C([0,T];F)} \leq \gamma(T) \|\varphi\|_{C([0,T];E)}$

(resp. $e^{\lambda A} * \varphi$ (5) $\in C(\bar{S}_{\theta,T}; F)$ for any $\varphi \in C(\bar{S}_{\theta,T}; E)$, and $\|e^{\lambda A} * \varphi\|_{C(\bar{S}_{\theta,T};F)} \leq \gamma(T) \|\varphi\|_{C(\bar{S}_{\theta,T};E)}$).

If $U \subset F$ is any open set, a function:

$$A : U \to L(F; E) \quad ; \quad y \to A(y)$$

is said to satisfy continuously the hypothesis $\mathscr{R}$ if for every $y \in U$, $A(y)$ satisfies the hypothesis $\mathscr{R}(\theta(y), M(y), \omega(y), c'(y), c''(y), \gamma(y))$ and the functions $\theta, M, \omega, c', c'' : U \to \mathbf{R}$, $\gamma : U \to C(\mathbf{R}^+; \mathbf{R})$ are continuous. □

From now on E and F will be real Banach spaces, F being densely and continuously imbedded in E.

(3) $L(F; E)$ is the Banach space of all linear continuous operators $A : F \to E$; we set $L(E) = L(E, E)$.

(4) If X is a Banach space, $C([0, T]; X)$ is the Banach space of all continuous mappings $u : [0, T] \to X$; $C^1([0, T]; X)$ is the Banach space of all continuously differentiable mappings $u : [0, T] \to X$.

(5) For any $g \in C([0, T]; E)$ set $e^{tA} * g(s) = \int_0^s e^{(s-\tau)A} g(\tau)\, d\tau$; for any $g \in C(\bar{S}_{\theta,T} E)$ set $e^{\lambda A} * g(z) = \int_0^z e^{(z-\xi)A} g(\xi)\, d\xi = \int_0^1 z e^{z(1-s)A} g(sz)\, ds$.

Let $\Omega \subset F$ be open and $f \in C^1(\Omega\,;E)$. Consider the following Cauchy problem:

$$(P)\begin{cases} u'(t) = f(u(t)) & t \geq 0 \\ u(0) = x & x \in \Omega\,. \end{cases}$$

THEOREM 2. *Let $f \in C^1(\Omega\,;E)$ and assume that $f'(x)$ satisfies continuously the hypothesis $\mathscr{R}$. Then for any $x_0 \in \Omega$ there exist: $r = r(x_0) > 0$, $r' = r'(x_0) > 0$, $T = T(x_0) > 0$ and a continuous function*:

$$u : [0\,,T] \times B(x_0\,,r) \to B(x_0\,,r')^{(6)} \quad ; \qquad (t\,,x) \to u(t\,,x)$$

such that $u(\cdot\,,x) \in C^1([0\,,T]\,;E) \cap C([0\,,T]\,;F)$ is the unique solution of (P) with values in $B(x_0\,,r')$.

Proof. For fixed $x_0\,,x \in \Omega\,,T > 0$, set

$$\psi_x(y) = f(y) - f(x) - f'(x)(y-x) \qquad \forall y \in \Omega.$$

Then (P) is equivalent to

$$(2) \qquad u(t) = x + \int_0^t e^{sf'(x)} f(x)\,ds + \int_0^t e^{(t-s)f'(x)}\,\psi_x(u(s))\,ds =$$

$$= v_x(t) + S_x(u)(t)$$

where

$$\begin{cases} v_x(t) = x + \displaystyle\int_0^t e^{sf'(x)} f(x)\,ds & \forall t \in [0\,,T] \\ S_x(w)(t) = \displaystyle\int_0^t e^{(t-s)f'(x)}\,\psi_x(w(s))\,ds & \forall w \in C([0\,,T]\,;F)\;\;,\;\; t \in [0\,,T]. \end{cases}$$

By the assumption $\mathscr{R}-(c)$, S_x maps $C([0\,,T]\,;F)$ into itself. Let us solve (2) by the Local Contractions Principle, in the ball $B(u_0\,,r)$ with $r > 0$ and $u_0(t) = x_0$ $\forall t \in [0\,,T]$.

Fix $\hat{T} > 0$, sufficiently large; then for every $w_1\,,w_2 \in B(u_0\,,r)$ we have:

$$\|S_x(w_1) - S_x(w_2)\|_{C([0,T];F)} \leq \sup_{T \in [0,\hat{T}]} \gamma(x)(T)\,\|\psi_x(w_1) - \psi_x(w_2)\|_{C([0,T];E)} \leq$$

$$\leq \sup_{T \in [0,\hat{T}]} \gamma(x)(T)\,[\|f(w_1) - f(w_2)\|_{C([0,T];E)} + \|f'(x)\|_{L(F;E)}\,\|w_1 - w_2\|_{C([0,T];F)}].$$

(6) If X is a Banach space, $x \in X\,,r > 0$, then $B(x\,,r) = \{y \in X\,;\|y-x\|_X \leq r\}$.

Let $r' = r'(x_0) \in]0, \operatorname{dist}(x_0, \partial\Omega)[$ be such that

$$\| S_x(w_1) - S_x(w_2) \|_{C([0,T];F)} \leq \tfrac{1}{2} \| w_1 - w_2 \|_{C([0,T];F)} \qquad \forall x \in B(x_0, r'),$$

By $\mathscr{R} - (a) - (b)$ we have, for fixed $t \in [0, T]$:

$$\| v_x(t) - x_0 + S_x(u_0)(t) \|_F \leq \| x - x_0 \|_F + \left\| \int_0^t e^{sf'(x)} (f(x) + \psi_x(x_0)) \, ds \right\|_F \leq$$

$$\leq \| x - x_0 \|_F + c''(x) [M(x) \, t e^{t\omega(x)} \| f(x) + \psi_x(x_0) \|_E +$$
$$+ \| (e^{tf'(x)} - 1)(f(x) + \psi_x(x_0)) \|_E] = \Phi(t, x).$$

Since Φ is continuous and $\Phi(0, x_0) = 0$, we may choose $T = T(x_0) \in]0, \hat{T}]$, $r = r(x_0) \in [0, r'[$ such that

$$\| v_x - u_0 + S_x(u_0) \|_{C([0,T];F)} \leq \frac{r'}{2} \qquad \forall x \in B(x_0, r).$$

The statement follows now applying the Principle of Local Contractions (see [3]) in the ball $B(u_0, r'(x_0)) \subset C([0, T]; F)$, since one can prove that, for any fixed $w \in C([0, T]; F)$, the functions

$$\begin{cases} \Omega \to C([0, T]; F); & x \to S_x(w) \\ \Omega \to C([0, T]; F); & x \to v_x \end{cases}$$

are continuous. □

For any fixed $x \in \Omega$ we can show, by classical arguments, global uniqueness of the solution of (P) and existence and uniqueness of the maximal solution (namely defined on a maximal time interval $J(x)$) of (P). Moreover $J(x)$ is relatively open in R^+.

THEOREM 3. *Under the hypotheses of Theorem 2, for every* $x \in \Omega$ *let* $u(\cdot, x): J(x) \to F$ *be the maximal solution of* (P). *Then, setting*:

$$\Lambda = \{(t, x) \; ; \; x \in \Omega \; , \; t \in J(x)\},$$

Λ is relatively open in $R^+ \times \Omega$ and $u: \Lambda \to F \; ; (t, x) \to u(t, x)$ is continuous.

Proof. Let $x_0 \in \Omega$, $0 < T < \tau(x) = \sup J(x)$. As the set $K = \{u(t, x_0); t \in [0, T]\}$ is compact, there exist $n \in N$, $y_1, \cdots, y_n \in K$ such that $\{\mathring{B}(y_i, r(y_i)) \; ; \; i = 1, \cdots, n\}$ [7] is a covering of K. Set $\overline{T} = \min\{T(y_i);$

(7) If X is a topological space and $A \subset X$, we denote by $\mathring{A}$ the interior of A.

$i = 1, \cdots, n\}$ (here $r(y_i)$ and $T(y_i)$ are defined as in Theorem 2). Since $x_0 \in \mathring{B}(y_{i_1}, r(y_{i_1}))$, there exists $s_1 > 0$ such that

$$u : [0, \overline{T}] \times B(x_0, s_1) \to F \; ; \qquad (t, x) \to u(t, x)$$

is continuous. Similarly, we have $u(\overline{T}, x_0) \in B(y_{i_2}, r(y_{i_2}))$ and hence there exists $s_2 > 0$ such that

$$u : [0, 2\overline{T}] \times B(x_0, s_2) \to F \; ; \qquad (t, x) \to u(t, x)$$

is continuous. Iterating this argument, after a finite number of steps we find $s > 0$ such that

$$u : [0, T] \times B(x_0, s) \to F \; ; \qquad (t, x) \to u(t, x)$$

is continuous; thus we show that for every $x_0 \in \Omega$ and $t \in J(x_0)$ there exists a neighbourhood of (t, x_0) where u is continuous. □

To prove the analyticity theorem we need the following lemma:

LEMMA 4. *Let* X , Y *be two complex Banach spaces,* Y *being densely and continuously imbedded in* X; *let* $U \subset Y$ *be an open set and let* $A : U \to L(Y ; X)$ *be a holomorphic mapping, satisfying continuously the hypothesis* $\mathcal{R}$. *Suppose that there exists* $\theta \in]0, \theta(y)]$ $\forall y \in U$. *Let* $g \in C(\overline{S}_{\theta,T} \times U ; X)$ *be holomorphic in* $S_{\theta,T} \times U$; *then the function* $u(\lambda, x) = (e^{zA(x)} * g(x))(\lambda)$ [5] *is the mild solution of the linear problem*:

$$\begin{cases} \dfrac{du}{d\lambda}(\lambda, x) = A(x)\, u(\lambda, x) + g(\lambda, x) & \lambda \in S_{\theta,T}. \\ u(0) = 0 . \end{cases}$$

Moreover u belongs to $C(\overline{S}_{\theta,T} \times U ; Y)$ and is holomorphic in $S_{\theta,T} \times U$, with values in Y. □

THEOREM 5. *Assume that* f *is defined and holomorphic, with values in* $E_{\mathbf{C}}$, *on a neighbourhood* Ω' *of* Ω *in* $F_{\mathbf{C}}$. *Let* $f'(x)$ *satisfy continuously the hypothesis* $\mathcal{R}$ *on* Ω'. *Then the function* $u : \Lambda \to F$, *defined as in Theorem 4, is real analytic.*

Proof. Let $x_0 \in \Omega'$, $\theta \in]0, \theta(x_0)[$; then there exists $\delta = \delta(x_0) \in]0$, dist $(x_0, \partial\Omega')[$ such that $S_{\theta(x)} \supset S_\theta$ for every $x \in B(x_0, \delta) \subset F_{\mathbf{C}}$. Let us consider the problem:

$$\begin{cases} u'(z) = f(u(z)) & z \in S_{\theta,\sigma} \; ; \; \sigma > 0 \\ u(0) = x & x \in B(x_0, \delta) \subset F_{\mathbf{C}}. \end{cases}$$

We may show, following the proof of Theorem 2, that there exist $\rho \in]0, \delta[$, $\rho', \sigma > 0$, depending on x_0, and a unique continuous function

$u : \overline{S}_{\theta,\sigma} \times B(x_0, \rho) \to B(x_0, \rho')$ which satisfies:

$$u(z, x) = x + \int_0^z e^{\lambda f'(x)} f(x) \, d\lambda + \int_0^z e^{(z-\lambda) f'(x)} \psi_x (u(\lambda)) \, d\lambda .$$

Let $\{u_n\}_{n \in N} : \overline{S}_{\theta,\sigma} \times B(x_0, \rho) \to B(x_0, \rho')$ be the approximating functions of u given by the Contractions Theorem:

$$u_0(z, x) = x$$

$$u_n(z, x) = x + \int_0^z e^{\lambda f'(x)} f(x) \, d\lambda + \int_0^z e^{(z-\lambda) f'(x)} \psi_x (u_{n-1}(\lambda)) \, d\lambda .$$

Clearly u_0 is holomorphic; moreover if u_{n-1} is assumed to be holomorphic in $S_{\theta,\sigma} \times \mathring{B}(x_0, \rho)$ [7], then it follows from Lemma 4 that also u_n is holomorphic in $S_{\theta,\sigma} \times \mathring{B}(x_0, \rho)$. Moreover $u_n \to u$ in $C(S_{\theta,\sigma} \times B(x_0, \rho) ; F_C)$, so that it can be shown that u is holomorphic in $S_{\theta,\sigma} \times \mathring{B}(x_0, \rho)$. In particular, if $x_0 \in \Omega$, then u is real analytic, with values in F, in $]0, \sigma[\times B(x_0, \rho)$. To accomplish the proof it now suffices to use the same arguments as in Theorem 3. □

References

[1] Cartan (1977) – *Cours de Calcul Différentiel. Hermann.*

[2] Da Prato and Grisvard (1979) – *Equations d'évolution abstraites non linéaires de type parabolique*, « Ann. di Mat. Pura ed Appl. », *120*, 329–396.

[3] Dieudonné (1960) – *Foundations of Modern Analysis*, vol. 1 – Academic Press, p. 260.

[4] Franzoni and Vesentini (1980) – *Holomorphic Maps and invariant distances.* North Holland.

[5] Tanabe (1979) – *Equations of Evolution.* Pitman.

[6] Yôsida (1974) – *Functional Analysis.* Springer Verlag.