Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Marshall J. Leitman, Epifanio G. Virga

On uniqueness for bounded channel flows of viscoelastic fluids

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **82** (1988), n.4, p. 717–723.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1988_8_82_4_717_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1988.

Atti Acc. Lincei Rend. fis.
(8), LXXXII (1988), pp. 717-723

Meccanica dei fluidi. — *On uniqueness for bounded channel flows of viscoelastic fluids.* Nota di Marshall J. Leitman e Epifanio G. Virga, presentata (*) dal Corrispondente T. Manacorda

Abstract. — It was conjectured in [1] that there is at most one bounded channel flow for a viscoelastic fluid whose stress relaxation function G is positive, integrable, and strictly convex. In this paper we prove the uniqueness of bounded channel flows, assuming G to be non-negative, integrable, and convex, but different from a very specific piecewise linear function. Furthermore, whenever these hypotheses apply, the unbounded channel flows, if any, must grow in time faster than any polynomial.

Key words: Uniqueness; Channel flows; Viscoelasticy fluids.

Riassunto. — *Sull'unicità di soluzione per l'equazione del moto in un canale di un fluido viscoelastico.* In [1] è stata avanzata la congettura che l'equazione che descrive il moto in un canale di un fluido viscoelastico la cui funzione di rilassamento degli sforzi G sia positiva, integrabile e strettamente convessa può avere al più una soluzione limitata. In questo lavoro l'unicità di soluzione è dimostrata assumendo che G sia non negativa, integrabile e convessa, ma diversa da una specialissima funzione lineare a tratti. Inoltre, quando ricorrono queste ipotesi, le eventuali soluzioni illimitate dell'equazione di moto devono divergere nel tempo più rapidamente di qualsiasi polinomio.

In [1] we considered the smooth bounded channel flows of a viscoelastic fluid. We sought solutions to the homogeneous equation of motion

$$\rho u_t(x,t) - \int_0^\infty G(s) u_{xx}(x, t-s)\,\mathrm{d}s = 0, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = u(\mathrm{L},t) = 0 \qquad \forall t \in \boldsymbol{R}, \tag{2}$$

in the class of functions (¹)

$$U := \left\{ u \in C^2(\Sigma) \,|\, 0 \leqq \sup_{\Sigma} |u| < \infty \right\},$$

where Σ is the strip

$$\Sigma := \{(x,t) : x \in [0,\mathrm{L}], t \in \boldsymbol{R}\}.$$

We denote by u the component of the velocity field along the axis of the channel. L is the width of the channel, ρ is a positive constant representing the mass density of the fluid per unit volume, and $G:(0,\infty) \to \boldsymbol{R}$ is the stress relaxation function.

In [1] we conjectured that *whenever the function G is positive, decreasing, integrable, and strictly convex, equations* (1) *and* (2) *have no solution in U except $u \equiv 0$.* Now we are able to prove more than that.

(*) Nella seduta del 22 giugno 1988.
(¹) The smoothness condition on $x \mapsto u(x,t)$ can be relaxed without difficulty; it is not central to our argument.

THEOREM. *If G is* (i) *non-negative,* (ii) *integrable, and* (iii) *convex, then $u \equiv 0$ is the only solution of* (1) *and* (2) *in U unless G is linear on each interval* $(kT_N, (k+1)T_N)$, $k = 0, 1, 2, \ldots$, *where*

$$T_N := \frac{2L}{N}\sqrt{\frac{\rho}{G(0)}} \tag{3}$$

and N is a positive integer.

As a consequence, we shall show that when G is the piecewise linear function described in the Theorem, (1) and (2) admit standing wave solutions of the form:

$$u(x,t) = \frac{1}{2i}\left[f\left(x + \sqrt{\frac{G(0)}{\rho}}\,t\right) - f\left(-x + \sqrt{\frac{G(0)}{\rho}}\,t\right)\right],$$

where f is any suitably smooth complex-valued $\left(\frac{2L}{N}\right)$-periodic function.

REMARK 1. The hypotheses (i)-(iii) imply that G is absolutely continuous on each closed sub-interval of $(0, \infty)$, and its derivative $\dot{G}$ is non-positive (almost everywhere). Of course $\lim_{t \to \infty} G(t) = 0$. Although redundant, the hypothesis that G be non-increasing is physically meaningful, and so it is frequently added to the list (i)-(iii).

REMARK 2. The Theorem asserts that, under hypotheses (i)-(iii), the solution to the non-homogeneous equation of motion for a viscoelastic fluid is unique within the class of smooth bounded flows, unless G is very special. Uniqueness theorems in viscoelasticity have also been proved within classes of functions vanishing asymptotically in the past (see *e.g.* [2] and [3]).

REMARK 3. If $G(t)$ or $\dot{G}(t)$, or both, become unbounded as t approaches zero, then G certainly cannot have the special piecewise linear form and, hence, the uniqueness referred to in Remark 2 obtains. Joseph, Renardy and Saut [4] have considered viscoelastic fluid responses of this type.

REMARK 4. As mentioned in [1] (Remark 4), if G has finite support some of the results presented here can also be obtained by methods of Hale [5].

The proof of the Theorem relies on the following Lemma.

LEMMA. *For every function $g\colon (0,\infty) \to \boldsymbol{R}$ that satisfies hypotheses* (i)-(iii) *of the Theorem, let $\hat{g}\colon \boldsymbol{R} \to \boldsymbol{C}$ be defined by*

$$\hat{g}(\lambda) := \int_0^\infty e^{-2\pi i \lambda t} g(t)\,\mathrm{d}t. \tag{4}$$

Then the function

$$\varphi(\lambda) := 2\pi i \lambda + \hat{g}(\lambda), \qquad \lambda \in \boldsymbol{R}, \tag{5}$$

has zeros if and only if g is piecewise linear with nodes equally spaced at intervals of $t_0 := 2\pi/\sqrt{g(0)}$. In this case $\varphi(\lambda)$ has precisely two simple zeros: $\lambda = \pm 1/t_0$.

PROOF. First observe that the complex equation $\varphi(\lambda)=0$ is equivalent to the two real equations:

$$\int_0^\infty \cos(2\pi\lambda t)g(t)\mathrm{d}t = 0, \tag{6}$$

$$\int_0^\infty \sin(2\pi\lambda t)g(t)\mathrm{d}t = 2\pi\lambda. \tag{7}$$

By (i) $\lambda=0$ is not a solution of (6). Moreover, if $\lambda_0 \neq 0$ is a solution of (6) and (7), so is $-\lambda_0$. Let $\lambda>0$. Since (iii) ensures that g is absolutely continuous on each closed subinterval of $(0,\infty)$ (see Remark 1), we can integrate by parts in the left-hand side of (6), and get

$$\int_0^\infty \sin(2\pi\lambda t)[-\dot{g}(t)]\mathrm{d}t = 0. \tag{8}$$

To see this, we must verify the formula (see Remark 3)

$$\int_0^\infty \cos(2\pi\lambda t)g(t)\mathrm{d}t = \int_0^\infty \frac{\sin(2\pi\lambda t)}{2\pi\lambda}[-\dot{g}(t)]\mathrm{d}t. \tag{9}$$

First observe that, for $0<\alpha<1/2\lambda<\beta<\infty$,

$$\int_\alpha^\beta \cos(2\pi\lambda t)g(t)\mathrm{d}t = -\int_\alpha^{1/2\lambda} \frac{\sin(2\pi\lambda t)}{2\pi\lambda}\dot{g}(t)\mathrm{d}t - \int_{1/2\lambda}^{\beta} \frac{\sin(2\pi\lambda t)}{2\pi\lambda}\dot{g}(t)\mathrm{d}t - $$
$$-\frac{\sin(2\pi\lambda\alpha)}{2\pi\lambda\alpha}\alpha g(\alpha) + \frac{\sin(2\pi\lambda\beta)}{2\pi\lambda}g(\beta).$$

By virtue of (i)-(iii) and Remark 1, we can let $\beta\to\infty$ in the latter formula, noting that the last term vanishes in the limit. The integral on the left-hand side exists as $\alpha\to 0$. Hence,

$$\lim_{\alpha\to 0}\left[\int_\alpha^{1/2\lambda} \frac{\sin(2\pi\lambda t)}{2\pi\lambda}[-\dot{g}(t)]\mathrm{d}t - \frac{\sin(2\pi\lambda\alpha)}{2\pi\lambda\alpha}\alpha g(\alpha)\right]$$

exists too. Now the limit of the integral is either non-negative or infinite, since the integrand is non-negative. If it is infinite, so is $\lim_{\alpha\to 0}\alpha g(\alpha)$, which violates (ii). Thus both terms remain finite as $\alpha\to 0$. In fact $\lim_{\alpha\to 0}\alpha g(\alpha)=0$, or else (ii) is again violated. Formula (9) is thus verified ([2]).

Since g satisfies (ii) and (iii), the function $t\mapsto -\dot{g}(t)$ is non-negative and non-increasing (almost everywhere). Thus $\lambda>0$ solves (8) only if $\dot{g}$ is constant (almost every-

([2]) Formula (9) remains valid if g is merely assumed to be non-negative, non-increasing, and integrable. In this case $-\dot{g}(t)\mathrm{d}t$ must be replaced by $\mathrm{d}\mu(t)$, where μ denotes the (non-negative) Borel measure induced by the function $-g$. The proof is the same.

where) on each interval

$$\left(\frac{k}{\lambda}, \frac{k+1}{\lambda}\right), \qquad k = 0, 1, 2, \ldots$$

Since g is absolutely continuous, it must have the form

$$g\left(\frac{k+s}{\lambda}\right) = (\gamma_{k+1} - \gamma_k)s + \gamma_k, \qquad 0 \leqq s \leqq 1, \qquad k = 0, 1, 2, \ldots,$$

where $\{\gamma_k\}_{k=0}^{\infty}$ is some non-negative, non-increasing, summable sequence. Substituting this g into (7) yields $(2\pi\lambda)^2 = g(0)$. Thus the zeros of (5) are

$$\lambda = \pm \frac{\sqrt{g(0)}}{2\pi}.$$

Finally, these roots are simple if and only if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \varphi\left(\frac{\sqrt{g(0)}}{2\pi} + \varepsilon\right) \neq 0.$$

A straightforward but tedious calculation shows that this limit is equal to $6\pi i$.

Proof of the Theorem. We extend u oddly in x to the interval $[-L, L]$ and compute its Fourier series in x:

$$u(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

where

$$v_n(t) := \frac{2}{L} \int_0^L u(x, t) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad n = 1, 2, \ldots.$$

If $u \in U$ solves (1) and (2), then each function v_n is bounded, of class $C^1(\boldsymbol{R})$, and solves the equation

$$\dot{v}_n(t) + \int_0^{\infty} g_n(s) v_n(t-s) ds = 0, \tag{10}$$

where

$$g_n(s) := \frac{1}{L}\left(\frac{n\pi}{\rho}\right)^2 G(s).$$

For a given n, we now seek solutions of (10) within the class of tempered distributions on $\boldsymbol{R}$ (3). If $h: \boldsymbol{R} \to \boldsymbol{C}$ is any integrable function we denote by $\hat{h}: \boldsymbol{R} \to \boldsymbol{C}$ its Fourier transform

$$\hat{h}(\lambda) := \int_{-\infty}^{+\infty} e^{-2\pi i \lambda t} h(t) dt.$$

If we extend G to $(-\infty, 0)$ by setting $G(t) = 0$ for every $t < 0$, then $\hat{g}_n$ is defined as in

(3) Bounded solutions of (10), if any, must be in $C^{\infty}(\boldsymbol{R})$. Indeed, if v_n is an everywhere differentiable function which solves (10) pointwise, a theorem of Leitman and Mizel (see [6], Sect. 4) shows that $\dot{v}_n$ is locally absolutely continuous. An induction argument completes the proof.

(4). Let $\hat{v}_n$ denote the Fourier transform of v_n regarded as a tempered distribution. Then (10) is equivalent to

$$\varphi_n \hat{v}_n = 0, \tag{11}$$

where φ_n is a function defined as in (5). The solutions of (11), if any, must have support on the set $\{\lambda \in \boldsymbol{R}: \varphi_n(\lambda) = 0\}$. Since g_n satisfies hypotheses (i)-(iii), the Lemma applies, and so $\hat{v}_n = 0$ is the only solution of (11), and $v_n = 0$ the only solution of (10), unless g_n is piecewise linear with nodes equally spaced at intervals of

$$T_n = \frac{2L}{n}\sqrt{\frac{\rho}{G(0)}}.$$

If this is the case, then the solutions of (10) within the class of tempered distributions are spanned by the bounded periodic functions [4]

$$t \mapsto e^{\pm 2\pi i t/T_n}. \tag{12}$$

If G is linear on each interval $(kT_N, (k+1)T_N)$, $k = 0, 1, 2, \ldots$, for some positive integer N, then every $g_{mN}, m = 1, 2, \ldots$, is also linear on each interval

$$\left(k\frac{T_N}{m}, (k+1)\frac{T_N}{m}\right),$$

and so can be regarded as a piecewise linear function with nodes equally spaced at intervals of T_{mN}. Thus $v_{mN}, m = 1, 2, \ldots$, are the only non-zero Fourier coefficients of u; they are periodic functions of the form (12) with T_n replaced by T_{mN}.

If G is not the piecewise linear function above, and satisfies (i)-(iii), then $u \equiv 0$ is the only solution of (1) and (2) in U, since every v_n vanishes.

REMARK 5. The argument employed to prove the Theorem also shows that whenever G satisfies (i)-(iii) any unbounded solution to (1) and (2) must grow faster than any polynomial as $|t| \to \infty$.

There are some consequences of the Theorem worth mentioning. Suppose that G satisfies (i)-(iii) and is *strictly convex* in a neighbourhood of some point. Then it cannot be piecewise linear, and the Theorem guarantees that the solutions of (1) and (2), if any, must be unbounded.

Now *fix* a positive integer N and write G_N for that piecewise linear interpolation of G with nodes equally spaced at intervals of T_N (the same as in (3)). Then the functions

$$u_N^{\pm}(x, t) = e^{\pm 2\pi i t/T_N} \sin\left(\frac{N\pi x}{L}\right),$$

solve (1) and (2) with G replaced by G_N. On the other hand, this G_N can also be regarded as a piecewise linear function with nodes equally spaced at T_N/m, for any posi-

(4) See footnote 3 above.

tive integer m. Since $T_N/m = T_{mN}$, it follows that the functions

$$u^{\pm}_{mN}(x,t) = e^{\pm 2\pi imt/T_N} \sin\left(\frac{mN\pi x}{L}\right) =$$

$$= \frac{1}{2i}\left[\exp\left(\frac{imN\pi}{L}\left(x \pm \sqrt{\frac{G(0)}{\rho}}\,t\right)\right) - \exp\left(\frac{imN\pi}{L}\left(-x \pm \sqrt{\frac{G(0)}{\rho}}\,t\right)\right)\right],$$

also solve (1) and (2) with G replaced by G_N. Furthermore, if $f\colon \boldsymbol{R} \to \boldsymbol{C}$ is any function with a sufficiently smooth Fourier series of the form:

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imN\pi x/L},$$

then

$$u_f(x,t) = \frac{1}{2i}\left[f\left(x + \sqrt{\frac{G(0)}{\rho}}\,t\right) - f\left(-x + \sqrt{\frac{G(0)}{\rho}}\,t\right)\right]$$

is again a bounded solution of (1) and (2) when G is replaced by G_N.

Recall that N was any fixed positive integer. By choosing N large enough, and hence T_N small enough, we can approximate G by G_N uniformly to any desired degree of accuracy. This means that if equations (1) and (2) do not admit globally bounded solutions for a given G, there are uniformly close approximations to G for which they do. We had already encountered in [1] a G for which this phenomenon occurs.

REMARK 6. The hypothesis that G be convex plays a crucial role in the proof of the Theorem. It is easy to show by example that if G is non-negative and integrable, but fails to be convex, then the Theorem is not true. Let

$$G(t) = \gamma t e^{-\alpha t} \quad \text{for } t \geqq 0, \tag{13}$$

with γ and α positive constants. Then (1) and (2) have solutions of the form

$$u(x,t) = u_0 \sin\left(\frac{p\pi x}{L}\right),$$

if

$$p = \sqrt{\frac{2L^2\alpha^3\rho}{\gamma\pi^2}}$$

is an integer.

The G in (13), while surely non-convex, also fails to be non-increasing. To see that it is just convexity which is at issue here, we can easily find a G which is non-negative, non-increasing, integrable but non convex, for which the situation is similar.

For example, let G have constant value on the interval $[0,\beta]$ and be zero on (β,∞). Then bounded solutions of (1) and (2) can be constructed as before, if and only if,

$$\frac{\beta}{\pi}\sqrt{2\alpha}$$

is an *odd* integer.

ACKNOWLEDGEMENT

This paper was written while M. J. Leitman was Visiting Professor at the Univeristy of Pisa. The support of the Italian C.N.R. is gratefully acknowledged.

REFERENCES

[1] M. J. LEITMAN and E. G. VIRGA (1988) – *On bounded channel flows of viscoelastic fluids, «Atti Acc. Lincei Rend. fis.», 82*, 291-297.
[2] G. CAPRIZ and E. G. VIRGA (1988) – *Un teorema di unicità in viscoelasticità lineare*, «Rend. Sem. Mat. Univ. Padova», 79, 15-24.
[3] G. VERGARA CAFFARELLI and E. G. VIRGA (1987) – *Sull'unicità della soluzione del problema dinamico della viscoelasticità lineare*, «Atti Acc. Lincei Rend. fis.», 81, 379-387.
[4] D. D. JOSEPH, M. RENARDY and J. C. SAUT (1984-85) – *Hyperbolicity and change of type in the flow of viscoelastic fluids*, «Arch. Rational Mech. Anal.», 87, 213-251.
[5] J. HALE (1971) – *Functional differential equations*, «Springer Verlag, Berlin», etc.
[6] M. J. LEITMAN and V. J. MIZEL (1974) – *On fading memory spaces and hereditary integral equations*, «Arch. Rational Mech. Anal.», 75, 18-51.