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Meccanica dei solidi. – *On linear versus nonlinear flow rules in strain localization analysis.* Nota di GIORGIO BORRÉ e GIULIO MAIER (*), presentata (**) dal Corrisp. G. MAIER.

ABSTRACT. – This note contains some remarks on the analysis of bifurcation phenomena, specifically strain localization (onset of a strain rate discontinuity), in small-deformation elastoplasticity. Nonassociative flow rules are allowed for to cover constitutive models frequently adopted for frictional (and softening) materials such as concrete. The conventional derivation of the localization criterion resting on an incrementally linear «comparison material» is critically reviewed and compared to the criterion resulting from «actual» nonlinear plastic flow laws. This communication anticipates, in an abbreviated form, results to be presented elsewhere in an extended form: therefore proofs of the propositions and various comments are omitted.

KEY WORDS: Plasticity; Localization; Softening.

RIASSUNTO. – *Sull'impiego di leggi di scorrimento lineari o nonlineari nei problemi di localizzazione in elastoplasticità.* Si svolgono alcune considerazioni sui fenomeni di biforcazione in solidi elastoplastici in regime di «piccole deformazioni» (di linearità geometrica) e precisamente sul manifestarsi di localizzazioni intese come discontinuità nel campo delle deformazioni incrementali.

Si considerano leggi nonassociate. Vengono così inclusi nella trattazione modelli costitutivi frequentemente adottati per descrivere il comportamento di materiali ad attrito interno e soggetti a danneggiamento (nel senso di degrado di rigidezze elastiche in seguito a deformazioni anelastiche), oltre che soggetti a manifestazioni di instabilità per incrudimento negativo («softening»). Si esamina criticamente il criterio, frequentemente adottato in letteratura, di localizzazione fondato su «materiale di confronto» incrementalmente lineare e si fornisce, corredato da osservazioni comparative, il criterio che risulta dall'assunzione del modello di materiale incrementalmente lineare e si fornisce, corredato da osservazioni comparative, il criterio che risulta dall'assunzione del modello di materiale incrementale non lineare. In questa nota alcuni risultati che saranno presentati altrove in forma più estesa e dettagliata vengono comunicati in forma abbreviata, omettendo le dimostrazioni e vari commenti.

1. INTRODUCTION

Strain localization phenomena in a number of technologically important categories of materials, have been the object of extensive investigations for years (e.g. [6] [11] [16] [17] [18] [19]), and are still attracting much attention in the recent

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literature, especially from the computational standpoint (e.g. [12] [13] [14] [20]).

The basic problem concerning localization can be concisely described as follows.

Consider an infinitesimal neighbourhood of a point in an elastic-plastic solid, or an elastic-plastic system (such as a material specimen in a testing machine) which is homogeneous as regards constitutive law, stress state and past yielding history. In the evolution of such a system the strain localization, understood as the appearance of a discontinuity in strain rates, marks the onset of a nonuniform response besides the uniform one; in other terms, it represents as bifurcation phenomenon. The evaluation of the thresholds at which such a phenomenon occurs and the determination of the main relevant kinematic and static features, represent basic questions for the nonlinear evolutive analysis up to failure of certain engineering systems, particularly of concrete, geomaterials and ceramics.

Bifurcation of the velocity gradient along a loading path can be caused by material destabilizing effects such as softening and lack of normality accomodated in the constitutive laws, even in the absence of geometrical destabilizing effects (i.e. large deformations affecting equilibrium equations). The notion of softening is originated by instability manifestations (understood as negative second order work performed by an external agency) observed in experiments on material specimens and structural elements in the absence of observable influence of geometry changes on equilibrium relations (see e.g. [1] [3]).

If elastic plastic constitutive laws are used in order to allow for softening in boundary value problems, far-reaching implications arise both in the theoretical framework and in the numerical solution procedures of such problems. These implications have been investigated by various Authors, sometimes compared to, and combined with, those due to lack of normality and damage by elastic-plastic coupling (e.g. [2] [10]) and to the implications of geometrical (instead of physical) destabilizing effects, which are due to large displacements and strains and are not accomodated in the material model chosen for simulation purposes (see e.g. [2] [8] [11]). Strain localization phenomena as defined earlier are among the possible consequences of softening and/or other destabilizing factors.

In this broad, partly still controversial mechanical context, this paper is intended to provide a contribution in the particular direction specified below.

Traditionally, in order to answer the above specified basic questions, the incremental nonlinear elastoplastic constitutive laws are replaced by fictitious linear flow rules, i.e. the «actual» materials is replaced by an incrementally linear «comparison material» (see e.g. [12] [13]).

Nonassociated flow laws were studied from the uniqueness and stability standpoints by Maier and Hueckel [8] [10], Raniecki and Bruhns [15] and Casey and Lee [2]. The second pair of Authors, allowing for finite deformations, established bounds to bracket the onset of the actual bifurcation on the basis of suitably chosen incrementally linear material model. Ortiz, Leroy and Needleman [11] developed a finite element method for localized failure analysis, resting on a preliminary solution (in each element and at each loading step) of the localization problem above referred to as « basic questions », such solution being obtained by adopting a linearized « comparison material » model.

This paper was motivated by the contributions of Ortiz et al. [12] [13] and aims at clarifying the links between the consequences of the adoption of an incrementally linear (instead of the original nonlinear) flow rule to localization analysis purposes, when the elastoplastic material model is of nonassociative type. Although before the completion of the present study a paper by B. Loret appeared containing a comprehensive investigation on the incremental nonlinearity in strain localization [7], the results presented here seem to be supplementary to, rather than overlapping on, Loret's results and to the earlier ones by Rice and Rudnicki [17] and by Chambon and Desrues [4].

2. FORMULATION OF THE PROBLEM

2.1. *Constitutive relations*

A nonassociative elastic-plastic constitutive law in incremental (rates) terms can be expressed in the following customary form around a stress state at yielding:

(1) $$\dot{\sigma}_{ij} = E_{ijhk}(\dot{\epsilon}_{hk} - \dot{\epsilon}^{p}_{hk})$$

(2) $$\dot{\epsilon}^{p}_{ij} = \frac{\partial \psi}{\partial \sigma_{ij}} \dot{\lambda}$$

(3) $$\dot{\lambda} \geqq 0$$

(4) $$\dot{\phi} = \frac{\partial \phi}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial \phi}{\partial \epsilon^{p}_{ij}} \dot{\epsilon}^{p}_{ij} \leqq 0$$

(5) $$\dot{\phi}\dot{\lambda} = 0$$

Here σ_{ij}, ϵ_{ij}, ϵ^{p}_{ij} denote the cartesian tensors of stresses, strains and plastic strains, respectively; ϕ and ψ the yield function and the plastic potential; $\dot{\lambda}$ is the plastic multiplier; E_{ijhk} is the elastic tensor endowed with the usual symmetries. The presence of « corners » on the yield surface has been excluded, i.e. the outward normal to it in the stress-point is uniquely defined.

Substitute eqs. (1), (2) into (4) and solve it with respect to $\dot{\lambda}$ having set $\dot{\phi} = 0$. Let the expression of $\dot{\lambda}$ thus obtained be substituted into eq. (2) and this into (1) and solve with respect to $\dot{\sigma}_{ij}$. These simple manipulations lead to the re-formulation which follows:

for « unloading » processes ($\dot{\phi} < 0$):

(6) $$\frac{\partial \phi}{\partial \sigma_{ij}} E_{ijhk} \dot{\epsilon}_{hk} < 0$$

$$\dot{\sigma}_{ij} = E_{ijhk}\dot{\epsilon}_{hk} \tag{7}$$

for «loading» processes ($\dot{\phi} = 0$):

$$\frac{\partial \phi}{\partial \sigma_{ij}} E_{ijhk}\dot{\epsilon}_{hk} \geqq 0 \tag{8}$$

$$\dot{\sigma}_{ij} = D_{ijhk}\dot{\epsilon}_{hk} \tag{9}$$

having set:

$$H \equiv -\frac{\partial \phi}{\partial \epsilon^{p}_{ij}} \frac{\partial \psi}{\partial \sigma_{ij}} \tag{10 a}$$

$$H_c \equiv -\frac{\partial \psi}{\partial \sigma_{ij}} E_{ijhk} \frac{\partial \phi}{\partial \sigma_{hk}} \tag{10 b}$$

$$P_{ijhk} \equiv \frac{1}{H - H_c} E_{ijpq} \frac{\partial \psi}{\partial \sigma_{pq}} \frac{\partial \phi}{\partial \sigma_{rs}} E_{rshk} \tag{11}$$

$$D_{ijhk} = E_{ijhk} - P_{ijhk} \tag{12}$$

The scalar H is referred to as hardening modulus and may be negative (softening material behaviour); H_c denotes the (negative) critical threshold of internal («snap-back») instability, which will be ruled out a priori in what follows. Eq. (12) defines the elastoplastic («tangent») stiffness matrix of the material.

2.2. *Relations governing the strain localization*

Let a strain rate discontinuity across a surface Γ with normal n_i be described in a point x_j as:

$$[\![\dot{\epsilon}_{ij}]\!] \equiv \dot{\epsilon}^{+}_{ij} - \dot{\epsilon}^{-}_{ij} \tag{13}$$

denoting by superscripts + and - the outside and inside faces of Γ with respect to direction n_i.
Maxwell kinematic condition (see e.g. [12]) reads:

$$\left[\!\!\left[\frac{\partial \dot{u}_i}{\partial x_j} \right]\!\!\right] = g_i n_j \tag{14}$$

where g_i is the vector which defines the discontinuity or «jump» in the velocity derivatives. Combining eqs (13) and (14) with and the geometric compatibility

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{15}$$

one obtains the kinematic equation which is central in the present context:

$$[\![\dot{\epsilon}_{ij}]\!] = \frac{1}{2}(g_i n_j + g_j n_i) \tag{16}$$

The special cases g_i normal to n_i («shear band») and g_i parallel to n_i («splitting mode») are worth noting.

The equilibrium across the discontinuity surface requires for the stress rate discontinuity:

$$n_i(\dot{\sigma}^+_{ij} - \dot{\sigma}_{ij}) = n_i [\![\dot{\sigma}_{ij}]\!] = 0 \tag{17}$$

We will combine below the constitutive relationships (6) - (9), the kinematic and static equations (16) and (17), for the three possible cases separately.

Case 1: unloading-unloading.

Eq. (7) entails:

$$[\![\dot{\sigma}_{ij}]\!] = E_{ijhk} [\![\dot{\epsilon}_{hk}]\!] \tag{18}$$

Substituting eq. (16) into (18) and eq. (18) into (17) we have:

$$A^e_{ij} g_j = 0, \qquad \text{having set } A^e_{ij} \equiv n_h E_{hijk} n_k \tag{19a,b}$$

Since matrix A_j^e is positive definite due to the properties of the elastic tensor E_{hijk}, no vector $g_j \neq 0$ solves eq. (19) and, hence, localization is impossible in this case.

Case 2: loading-loading.

Eq. (9) implies:

$$[\![\dot{\sigma}_{ij}]\!] = D_{ijhk} [\![\dot{\epsilon}_{hk}]\!] \tag{20}$$

As is case 1, substituting eq. (16) into (19) and eq. (19) into (17), we obtain from (20):

$$A_{ij} g_j = 0, \qquad \text{where: } A_{ij} \equiv n_h D_{hijk} n_k \tag{21a,b}$$

The «localization matrix» A_{ij} is not necessarily positive definite. Therefore localization may occur in this case provided some n_i and $g_i \neq 0$ satisfy eq. (21) and simultaneously the kinematic equation (16) and the constitutive inequalities (8). These are rewritten below for convenience:

$$\frac{\partial \phi}{\partial \sigma_{ij}} E_{ijhk} \dot{\epsilon}^+_{hk} \geqq 0 \tag{22a,b}$$

$$\frac{\partial \phi}{\partial \sigma_{ij}} E_{ijhk} (\dot{\epsilon}^+_{hk} - [\![\dot{\epsilon}_{hk}]\!]) \geqq 0$$

Case 3: loading (on face $+$) - unloading (on face $-$).

The constitutive eqs. (7) and (9) combined with (13), require that:

$$[\![\dot{\sigma}_{ij}]\!] = D_{ijhk}\dot{\epsilon}^{+}_{hk} - E_{ijhk}(\dot{\epsilon}^{+}_{hk} - [\![\dot{\epsilon}_{hk}]\!]) \tag{23}$$

Substitute eq. (16) into (23) and this into (17) to obtain the following nonhomogeneous counterpart to the homogeneous eqs. (19) and (21) of cases 1 and 2:

$$A^{e}_{ij} g_j = n_h P_{hijk} \dot{\epsilon}^{+}_{ik} \tag{24}$$

In this case, bifurcation occurs if some vectors n_i, $g_i \neq 0$ solve eq. (24) and simultaneously satisfy the kinematic eq. (16) and the constitutive inequalities (8) and (6). These inequalities are:

$$\frac{\partial \phi}{\partial \sigma_{ij}} E_{ijhk}\dot{\epsilon}^{+}_{hk} \geqq 0; \tag{25a,b}$$

$$\frac{\partial \phi}{\partial \sigma_{ij}} E_{ijhk}(\dot{\epsilon}^{+}_{hk} - [\![\dot{\epsilon}_{hk}]\!] 0) \leqq 0$$

2.3. *Localization problems*

On the basis of what precedes, the following basic questions will be addressed in the subsequent Sections. These questions have been examined earlier e.g. by Rice et al. in [16] [17] [18].

Problem A: For given stress state and past yielding history, and, hence, in particular for a given hardening (softening) modulus H (eq. (10a)), can strain localization occur and which are the relevant discontinuity surface n_i and vector g_i?

Problem B: Since the hardening modulus H can be regarded as the main discriminant quantity for the present bifurcation phenomenon, what is the threshold $H^* \equiv \inf\{H'\}$, H' being all moduli for which localization is impossible and the only possible incremental response preserves homogeneity?

In order to simplify the notation, we set:

$$\mathbf{A} = [n_h D_{hijk} n_k]; \qquad \mathbf{n} = \{n_i\}; \qquad \mathbf{g} = \{g_i\} \tag{26}$$

3. On localization criteria for given hardening (or softening) modulus.

Using the conventional preliminary anlysis carried out in Sec. 2 we state here some propositions concerning the localization analysis for given hardening (softening) modulus H (problem A). Mechanical implications and interpretations will be discussed in the extended version where also formal proofs will be provided.

Proposition 1. (see e.g. [12]) Any pair of vectors $\mathbf{n}$ and $\mathbf{g} \neq 0$, such that $\mathbf{A}(\mathbf{n})\,\mathbf{g} = 0$, satisfies all other conditions (22) pertaining to the loading-loading case.

Proposition 2. In the loading-unloading case the condition $\det \mathbf{A}(\mathbf{n}) \leqslant 0$ is sufficient for bifurcation.

Proposition 3. In the loading-unloading case, the condition $\det \mathbf{A}(\mathbf{n}) \leqq 0$ is necessary for bifurcation.

Proposition 4. In the strain localization problem (*A*) for given hardening or softening modulus H, if both loading-unloading and loading-loading cases are allowed for (i.e. « real » nonlinear incremental flow rules are adopted instead of the « linear comparison material ») the criterion $\det \mathbf{A}(\mathbf{n}) \leqslant 0$ represents a sufficient and necessary condition for bifurcation.

This statement merely condenses propositions 1, 2 and 3.

4. Determination of hardening (softening) ranges for which bifurcation occurs

Now let the hardening modulus H be regarded as variable, though still assumed to be larger than the internal instability (snap-back) onset value H_C. Then the following statements can be proved.

Proposition 5. According to the bifurcation criterion $\det \mathbf{A}(\mathbf{n}) \leqq 0$ (for incrementally nonlinear materials), localization is possible for any modulus H such that $H_c < H \leqq H^*$ (« localization range »), where H (« localization onset modulus ») defined by eq. (53) is the infimum of the range where localization is not possible. In other terms, the set of H values for which localization is possible forms the (simply connected) interval $\overset{\rightharpoondown}{H_C H^*}$.

Proposition 6. The localization onset modulus H^* can be equivalently determined either by the criterion $\det \mathbf{A}(\mathbf{n}) \leqq 0$ (incrementally nonlinear constitution) or by the criterion $\det \mathbf{A}(\mathbf{n}) = 0$ (linear comparison material); in other terms, for $H = H^*$, $\det \mathbf{A}(\mathbf{n}) = 0$ for some $\bar{\mathbf{n}}$ and non negative for all other directions $\mathbf{n}$.

5. Closing remarks

The conclusions of the foregoing problem formulation (Sec. 2) and propositions (Secs. 3 and 4) can be summarized and commented upon as follows.

(a) The nonpositiveness of the determinant of the localization matrix $\mathbf{A}(\mathbf{n})$ constructed with the tangent elastic-plastic tensor and the discontinuity direction n, has been proved (in Sec. 3) to provide a sufficient and necessary bifurcation criterion for the « real » nonlinear nonassociative flow rules.

According to Ortiz et al. [12], the determination of the normal **n** defining the discontinuity plane in the incrementally linear (comparison material) approach is numerically performed by minimizing det $\mathbf{A}(\mathbf{n})$ with respect to **n**. This procedure gives discontinuity directions **n** for which the determinant of the localization matrix **A** is in general negative and, hence, violates the condition det $\mathbf{A}(\mathbf{n}) = 0$. Such a violation is usually accepted assuming that det $\mathbf{A}(\mathbf{n})$ has reached negative values as a consequence of a continuous variation from positive values. This implies a transition through the zero, which, however, cannot be detected by a customary approximate time-integration, algorithm due to the finiteness of the time step.

On the other hand, if the incremental nonlinearity is taken into account, the set of discontinuity directions n enabling localization is widened up to include all normals n for which det $\mathbf{A}(\mathbf{n}) < 0$. The present approach and resulting criterion det $\mathbf{A}(\mathbf{n}) \leqslant 0$ does not require a continuous variation of det $\mathbf{A}(\mathbf{n})$, in order to legitimate the numerical procedure discussed above.

The above distinction is worth noticing when the modulus H exhibits a discontinuity (« jump ») along the plastic evolution, as it does for some material models (e.g. with piecewise linear hardening). This circumstance becomes self-evident in the uniaxial stress case of a tensile uniform bar model of a material whose behaviour is described by a softening stress-strain constitution.

(b) When incremental nonlinearity is assumed, the condition det $\mathbf{A}(\mathbf{n}) = 0$ (in Sec. 4) singles out the critical value H^*, as it does under the stronger assumption of incremental linearity (H^* denotes the maximum hardening modulus for which bifurcation occurs). By the path of reasoning adopted in the present note, the bifurcation for $H < H^*$ is demonstrated to occur, in the sense that the existence of a nonuniform solution besides the uniform one is guaranteed. This conclusion includes the one reached by Rice and Rudnicki [17] by a different path of reasoning for incremental nonlinearity; however, although differing from their approach, the criterion det $\mathbf{A}(\mathbf{n}) \leqq 0$ turns out to apply also in the transition case $H = H^*$.

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