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Breathers for nonlinear wave equations

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Equazioni differenziali ordinarie. – *Breathers for nonlinear wave equations.* Nota di MICHAEL W. SMILEY, presentata (*) dal Socio Straniero L. CESARI.

ABSTRACT. – The semilinear differential equation (1), (2), (3), in $\mathbb{R} \times \Omega$ with $\Omega \subset \mathbb{R}^N$, (a nonlinear wave equation) is studied. In particular for $\Omega = \mathbb{R}^3$, the existence is shown of a weak solution $u(t,x)$, periodic with period T, non-constant with respect to t, and radially symmetric in the spatial variables, that is of the form $u(t,x) = \nu(t, |x|)$. The proof is based on a distributional interpretation for a linear equation corresponding to the given problem, on the Paley-Wiener criterion for the Laplace Transform, and on the alternative method of Cesari.

KEY WORDS: Periodicity; Breathers; Distributional solution; Weighted Hilbert space; Method of alternative problems.

RIASSUNTO. – *Soluzioni libere per equazioni delle onde non lineari.* Si studiano equazioni differenziali semilineari (1), (2), (3), in $\mathbb{R} \times \Omega$ con $\Omega \subset \mathbb{R}^N$ (equazioni delle onde nonlineari). In particolare, per $\Omega = \mathbb{R}^3$, si dimostra l'esistenza di soluzioni $u(t,x)$ deboli, periodiche di periodo T, non costanti rispetto a t, e radiali nelle variabili spaziali, cioè della forma $u(t,x) = U(t,|x|)$. La dimostrazione è basata su una interpretazione distribuzionale di una equazione lineare corrispondente al problema dato, sul criterio di Paley-Wiener per la trasformazione di Laplace, e sul metodo alternativo di Cesari.

The study of periodic phenomena has a long and distinguished history which spans many disciplines. From a mathematical perspective, a description of periodic behaviour may be incorporated into a model in the form of a boundary condition. An example of this is the problem of time periodic waves

$$u_{tt} - \Delta u + g(u) = f, \qquad (t,x) \in \mathbb{R} \times \Omega, \tag{1}$$

$$u(t,x) = 0, \qquad (t,x) \in \mathbb{R} \times \partial\Omega, \tag{2}$$

$$u(t+T,x) = u(t,x), \qquad (t,x) \in \mathbb{R} \times \Omega. \tag{3}$$

The first equation is a semilinear partial differential equation (a nonlinear wave equation) in which $\Delta u = \operatorname{div}(\operatorname{grad}(u))$, $g: \mathbb{R} \to \mathbb{R}$, and the non-homogeneous forcing term f is assumed T-periodic in t. The periodicity of u is prescribed in (3), while behaviour at the boundary of the spatial domain $\Omega \subset \mathbb{R}^N$ is prescribed in (2). This problem is said to be ill-posed since, in general, it is only when (3) is replaced by the initial con-

(*) Nella seduta del 14 maggio 1988.

ditions $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$ that solutions of the wave problem are known to be unique. Thus the periodic wave problem has an unusual character, and requires different methods of analysis than the problem of wave propagation from an initial state.

Suppose that f is identically zero in (1). The problem is then said to be homogeneous. From inspection of (1)-(3) it is clear that the homogeneous problem always has the trivial solution, $u(t,x) = 0$ for $(t,x) \in \mathbb{R} \times \Omega$. It may happen that there are other solutions as well, and in this case uniqueness of solutions certainly fails. If there are other solutions, which are truly time-dependent, it is customary to call them breathers. This term is meant to be descriptive of the undulation of the spatial profile of a solution, $\{(x, u(t,x)) : x \in \Omega\}$, that would be observed if viewed over an interval of time. The question of whether breathers exist for (1)-(3) is of considerable interest and has been addressed by several authors. The purpose of this note is to communicate some new results regarding this problem in the case where $\Omega = \mathbb{R}^3$ and the solutions are radially symmetric. With this choice of Ω, equation (2) is more properly stated in the form of a limit. By radial symmetry we mean that $u(t,x) = U(t,r)$, where $r = |x|$, for some $U: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$.

Our investigations of problem (1)-(3) were initiated under the guidance of Professor Lamberto Cesari, at the University of Michigan. Subsequent work lead to a theory [5] that is applicable to the case of bounded domains Ω. Professor Cesari, who is well-known for his work on periodic solutions of differential equations, is also a major contributor in the area of nonlinear functional analysis. Through his work, he continues to influence the author's work. In particular it has been the method of alternative problems, which is described in [1], that has provided the basis for the results presented here.

Weak solutions

The type of solutions being considered is that of radially symmetric T-periodic distributions. More precisely, let $\mathbb{R}^+ = (0, +\infty)$ and $C_0^\infty(\mathbb{R}^+)$ denote the set of C^∞ functions having compact support in $\mathbb{R}^+$. We define the set of test functions to be $D_T = \{\varphi \in C^\infty(\mathbb{R} \times \mathbb{R}^+) : \varphi(t+T, r) = \varphi(t,r)$ for all $(t,r) \in \mathbb{R} \times \mathbb{R}^+$, and $\varphi(t,\cdot) \in C_0^\infty(\mathbb{R}^+)$ for all $t \in \mathbb{R}\}$. Assuming that $\Omega = \mathbb{R}^3$ and u,f are radially symmetric, we change variables in (1), setting $w = ru$ and $h = rf$ where $r = |x|$, to obtain

$$w_{tt} - w_{rr} + rg(w/r) = h, \qquad (t,r) \in \mathbb{R} \times \mathbb{R}^+. \tag{4}$$

Clearly $w, h \in L^2((0,T) \times \mathbb{R}^+)$ is equivalent to $u, f \in L^2((0,T) \times \mathbb{R}^3)$. In addition to (4), and according to our weak interpretation of (2)-(3), $w(t,r)$ should also satisfy

$$w(t,\cdot) \in L^2(\mathbb{R}^+) \qquad \text{(a.e.)}\, t \in \mathbb{R}, \tag{5}$$

$$w(t+T, r) = w(t,r) \qquad \text{(a.e.)}\,(t,r) \in \mathbb{R} \times \mathbb{R}^+. \tag{6}$$

We say that w is a solution of (4)-(6), and that the corresponding u is a solution of (1)-(3), if $w: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfies: i) $w \in L^2((0,T) \times \mathbb{R}^+)$, ii) $w(t+T,r) = w(t,r)$ (a.e.) $(t,r) \in \mathbb{R} \times \mathbb{R}^+$ and iii) for all $\varphi \in D_T$ one has

$$\int_0^T \int_0^{+\infty} \{w[\varphi_{tt} - \varphi_{rr}] + [rg(w/r) - h]\varphi\} dr\, dt = 0 \tag{7}$$

Observe that (2) is now being interpreted as the condition: $u(t,\cdot) \in L^2(\mathbb{R}^3)$, (a.e.) $t \in \mathbb{R}^3$, so that the limiting behavior is only weakly required. Despite this apparently weak form of (2) we find that solutions have exponential decay at infinity. Functions of this type are said to be localized in space. We also point out that the first condition in (5) represents a weak regularity condition at the origin. By the nature of the test functions in D_T, (7) checks only that the differential equation is satisfied away from the origin. Thus a solution is only required to be square-integrable in a neighbourhood of the origin.

BREATHERS AND FORCED OSCILLATIONS

We use $H_{T,\delta}$ to denote the weighted Hilbert space of functions $w: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ which are T-periodic in t and have finite norm

$$\|w\|_{T,\delta} = \left\{\frac{1}{T} \int_0^T \int_0^{+\infty} |w(t,r)|^2 e^{2\delta r} dr\, dt\right\}^{1/2} < +\infty. \tag{8}$$

An associated weighted Sobolev space is $H^1_{T,\delta} = \{w \in H_{T,\delta}: w_t, w_r \in H_{T,\delta}\}$. The derivatives are distributional derivatives relative to D_T. The norm in $H^1_{T,\delta}$ is the usual Hilbert-Sobolev norm, weighted as in (8). Appropriate choices for δ will be given below.

The numbers $\theta_n = 2n\pi/T$, for integers $n \geq 0$, play an important role in describing the solution set of (1)-(3) when $\Omega = \mathbb{R}^3$. It is well known that the numbers $\{\theta_n^2\}_{n=0}^{\infty}$ are the eigenvalues for the linear operator consisting of ∂_t^2 together with T-periodic boundary conditions. The significance of one of these numbers for all space problems was first addressed by J.M. Coron [2]. He showed that when $\Omega = \mathbb{R}$, breathers could not exist if $g'(0) < \theta_1^2$. Subsequently, H.A. Levine [3] showed that, when $\Omega = \mathbb{R}^N - \{0\}$ and solutions were radially symmetric, the same inequality implied non-existence of breathers. In our work we show that all the numbers θ_n^2 have significance with regard to the nature of the set of solutions.

For convenience of notation let $\lambda = g'(0)$ and $\theta_{-1}^2 = -\infty$. Let $m = m_\lambda \geq -1$ be the integer such that $\theta_m^2 < \lambda \leq \theta_{m+1}^2$, and set $\beta_m = (\lambda - \theta_m^2)^{1/2}$ with the convention that $\beta_{-1} = +\infty$. We choose the weight function constant δ so that $0 < \delta < \beta_m$. With these notations established we can now describe the set breathers of (4)-(6), and hence the radially symmetric breathers of (1)-(3). A proof can be found in [7].

THEOREM. *Suppose h is the zero function in* (4). *If* $m_\lambda \geq 1$ *then there are infinitely many solutions* $w \in H^1_{T,\delta}$ *of* (4)-(6). *Local to the origin in* $H^1_{T,\delta}$, *all solutions can be characterized as lying on a* $(2m + 1)$*-dimensional manifold. This manifold has a* $2m$*-dimensional submanifold of breathers; the remaining* 1*-dimensional slice consists of time-independent solutions.*

REMARKS

1) The above result remains true as stated with $\Omega = \mathbb{R}^1$.

2) The above result generalizes to non-homogeneous problems, in the sense that there is a neighbourhood of the origin $B \subset H_{T,\delta}$ such that for all $h \in B$ the theorem is true.

3) When $m_\lambda = -1$ there is (locally) only the trivial solution, while for $m_\lambda = 0$ there is (locally) a 1-dimensional manifold of time-independent solutions.

4) The solutions depend continuously on $\lambda = g'(0)$, and on $h \in B$ if the problem is non-homogeneous.

FURTHER REMARKS

As suggested by the statement of the theorem, the results we have obtained are of a local nature. Essentially we linearize about the trivial solution, and then use an implicit function theorem argument. There are three main ingredients from functional analysis which are used. First the linearized problem is considered. This problem consists of

$$(9) \qquad Lw = w_{tt} - w_{rr} + \lambda w = h, \qquad (t,r) \in \mathbb{R} \times \mathbb{R}^+$$

together with (5)-(6), as understood in the distributional sense described above. In particular the decay condition as $r \to +\infty$ is only enforced by the requirement $w(t, \cdot) \in L^2(\mathbb{R}^+)$, (a.e.) $t \in \mathbb{R}$. Using the classical criterion of Paley-Wiener [4], given in terms of the Laplace transform, we show [6] that the linear operator L is an (unbounded) Fredholm operator of index $2m_\lambda + 1$. The second ingredient is the method of alternative problems [1]. Using our knowledge of the linear problem we recast the problem (4)-(6) as a parameterized operator equation. This equation is called the auxiliary equation in the terminology of the alternative method. We then use the third ingredient, the contraction mapping principle to show (locally) the auxiliary equation has solutions for all values of the parameter u_0. This parameter represents an arbitrary function in the null space of L and is used in the manifold parameterization. The contraction mapping theorem is used due to the fact that the implicit function theorem, as it is normally stated, cannot be used when $\Omega = \mathbb{R}^3$. This technical difficulty arises since the non-linearity $rg(w/r)$ is not (Frèchet) differentiable at the origin.

The boundedness and asymptotic decay of the solutions w of (4)-(6) are guaranteed by an estimate of the form

$$|w(t,r)| \leq C\exp(-\delta r), \qquad \text{(a.e.)} \qquad (t,r) \in \mathbb{R} \times \mathbb{R}^+. \tag{10}$$

This is ultimately due to a similar estimate available for the linear problem. However, the radial solutions $u(t,x) = w(t,r)/r$ may be of order $O(r^{-1})$ as $r \to 0^+$. It remains to be shown that there are some special solutions $u(t,x)$ which are bounded at the origin. Regardless of the behaviour at the origin these solutions are of order $O(r^{-1}\exp(-\delta r))$ as $r \to +\infty$, and thus are localized in space.

There is obviously an unusual bifurcation in the solution set as λ crosses each of the values θ_n. Clearly it is not just a finite count of solutions that is changing as λ crosses these values, but rather the dimensional count of the manifold of solutions. From Theorem I it is apparent that the bifurcation is from infinity, at least relative to the $H^1_{T,\delta}$ norm.

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