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A generalization to nonlinear hardening of the first shakedown theorem for discrete elastic-plastic structural models

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Meccanica dei solidi e delle strutture. — *A generalization to nonlinear hardening of the first shakedown theorem for discrete elastic-plastic structural models* (*). Nota (**) del Corrisp. GIULIO MAIER (***).

ABSTRACT. — In the plastic constitutive laws the yield functions are assumed to be linear in the stresses, but generally non-linear in the internal variables which are non-decreasing measures of the contribution to plastic strains by each face of the yield surface. The structural models referred to for simplicity are aggregates of constant-strain finite elements. Influence of geometry changes on equilibrium are allowed for in a linearized way (the equilibrium equation contains a bilinear term in the displacements and pre-existing stresses).

It is shown that shakedown (which means plastic work bounded in time) is guaranteed under variable-repeated quasi-static external actions, when the hardening behaviour exhibits reciprocal interaction, a suitably defined energy function of the internal variables is convex and the yield conditions can be satisfied at any time by some constant internal variable vector and by the linear elastic stress response.

Some interpretations and extensions of this result are envisaged. By specialization to linear hardening, earlier results are recovered, which reduce to Melan's classical theorem for non-hardening (perfectly plastic) cases.

KEY WORDS: Plasticity; Shakedown; Hardening

RIASSUNTO. — *Una generalizzazione all'incrudimento nonlineare del primo teorema di adattamento per modelli discreti di strutture elastoplastiche.* Nelle leggi costitutive plastiche qui considerate le funzioni di snervamento (o potenziali plastici) sono assunte lineari nelle tensioni e genericamente non lineari nelle variabili interne (nondecrescenti) che rappresentano misure del contributo alla deformazione di ciascuna faccia del poliedro che definisce il dominio elastico istantaneo nello spazio delle tensioni. I modelli strutturali discreti a cui si fa riferimento per semplicità sono aggregati di elementi finiti a spostamento lineare. L'influenza dei cambiamenti di configurazione sull'equilibrio è tenuta in conto in forma linearizzata (con un termine lineare negli spostamenti e negli sforzi preesistenti).

Si dimostra che l'adattamento o « shakedown » nella risposta ad azioni esterne variabili ripetute quasi-statiche, è assicurato sotto le condizioni che l'incrudimento presenti interazione reciproca, una opportuna funzione energia sia convessa nelle variabili interne e che le condizioni di plasticità siano soddisfatte ad ogni istante da un vettore costante di variabili interne e dalla risposta tensionale elastica lineare. Si accenna a qualche conseguenza, interpretazione ed estensione di questo risultato. Particolarizzando all'incrudimento lineare si ritrovano risultati precedenti che si riducono al classico teorema di Melan nei casi di plasticità perfetta.

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1. Introduction

The analysis of the inelastic response of structures to variable-repeated (in particular cyclic) external actions (such as forces and temperature changes) is important and sometimes crucial in various areas of engineering, expecially of nuclear and ocean technologies.

Often the evolution in time of the external actions (briefly, the "loading history") can be foreseen a priori only as for the variation ranges of the governing parameters ("load domain"). The material behaviour can frequently be described as linearly elastic, perfectly plastic (hence, time-independent or inviscid) and stable in Drucker's sense (which implies convexity of the yield surface and "associative" flow rule, i.e. normality of the plastic strain rate vector to this surface). Then the classical shakedown theory provides both illuminating qualitative interpretations of the overall mechanical behaviour and rational analysis tools for achieving quantitative information needed for design purposes.

Shakedown or "adaptation" means boundedness in time of the plastic deformations (or at least of the total energy dissipation) and represents a necessary, though not sufficient, condition for the safety of any plastic system subjected to variable-repeated loads. The classical shakedown theory centred on the Bleich-Melan theorem, and the many developments occurred in the last few decades have been the subject of comprehensive surveys and unified presentations, see e.g. references [1], [2], [3], [4]. Extensions to dynamics, pionereed by G. Ceradini, were surveyed in [5].

Referring to discrete (finite element) structural models and to piecewise-linearized yield surfaces, the writer developed in [6], [7] and in subsequent papers a shakedown theory allowing for linear hardening. This generalization has some theoretical and practical interest for the assessment of safety factors with respect to lack of shakedown in the presence of geometric effects (a trade-off arises between such stabilizing or instabilizing effects and hardening or softening) and especially for the determination of bounds on post-shakedown and, more generally, history-dependent quantities [7]. Several noteworthy results have been achieved in recent times by L. Corradi, J.A. König, J.B. Martin, C. Polizzotto, A.R.S. Ponter, F.S.K. Tin Loi and other Authors, mostly in the aforementioned framework of piecewise-linear yield surfaces, linear hardening or non-hardening behaviour and no influence of displacements on equilibrium equations. D. Weichert contributed to the generalization of the theory to large displacements with non-hardening constitution.

However, hardening understood as dependence of yield limits on plastic strains, when it is an essential feature of local plastic behaviour, is always nonlinear and its simulation in a piecewise linear way is cumbersome and computationally inefficient, as the number of yield modes is drastically increased.

In this paper general non-linear, instead of linear, hardening (or softening) behaviour is allowed for, in association with piecewise-linear yield surfaces, discrete structural models and linearized ("second order") geometric effects on equilibrium. Conditions are determined under which shakedown is guaranteed, thus further generalizing the main conclusion of ref. [7], i.e. the extension of the first (Melan's) static shakedown theorems to linear hardening and geometric effects. Parallel generalizations of other parts of shakedown theory (second, Koiter's theorem; Ceradini's dynamic shakedown theorem; various bounds on history-dependent quantities) are only envisaged here and will be pursued elsewhere.

2. AN INTERNAL VARIABLE DESCRIPTION OF THE PLASTIC BEHAVIOUR OF ELEMENTS

The following nomenclature will frequently be used in this paper. Underlined symbols denote matrices and column-vectors; vector inequalities apply componentwise; **0** is a matrix or vector with all zero entries; a tilde means transpose. A dot denotes time derivative, time t being an ordering variable, i.e. any increasing function of the physical time (this arbitrariness reflects the inviscid or time-independent and quasi-static nature of the system considered). Superscripts e and p mark linear elastic responses to (given) external actions and (unknown) plastic strains, respectively; $\forall$ means "for all"; finally, vectors $\boldsymbol{q}$, $\boldsymbol{e}$, $\boldsymbol{p}$ and $\mathbf{Q}$ will collect, in the same order, all components of total, elastic, plastic strain and stress, respectively (in this Sec., for a single structural constituent or finite element; subsequently, for all the constituents or elements in the whole structure or discretized structural model). Other symbols will be defined when they are employed first.

Strains $\boldsymbol{q}$ and stresses $\mathbf{Q}$ are understood here in Prager's generalized sense, namely: the scalar product of their vectors represents virtual work in the element concerned; they are intrinsic, "natural" variables, invariant with respect to rigid body motions. In order to make the subsequent developments more explicit without formal complications, two particular kinds of discrete constituents are worth being referred to: (*a*) the truss member (straight pin-ended bar), for which $\boldsymbol{q}$ and $\mathbf{Q}$ reduce to scalars (elongation and axial force, respectively); (*b*) the constant-strain, homogeneous, finite element (four-nodes tetrahedron) for three-dimensional continua, where $\boldsymbol{q}$ is the 6-component vector of edge elongations and $\mathbf{Q}$ defines the corresponding, self-equilibrated nodal forces acting in pairs along the edges; (*c*) the constant strain, homogeneous, three-nodes finite element in plane stress, $\boldsymbol{q}$ and $\mathbf{Q}$ being 3-vectors of its "natural" variables. In case (*a*) the affine transformation which relates the bar Q vs. q law to the material $\boldsymbol{\sigma}$ vs. $\boldsymbol{\varepsilon}$ law in uniaxial stress states, is self-evident.

In case (*b*) it is worth noting that, similarly, the $\mathbf{Q}$ sv. $\boldsymbol{q}$ relationship reflects the $\boldsymbol{\sigma}$ vs. $\boldsymbol{\varepsilon}$ material constitution in triaxial stress states. In fact, let the 6-vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ collect the independent components of the Cauchy stress tensor and the actual strains (with "engineering definition" shear strains); the we have:

$$\boldsymbol{q} = \mathbf{T}\boldsymbol{\varepsilon} \quad , \quad \boldsymbol{\sigma} = \frac{1}{V}\tilde{\mathbf{T}}\mathbf{Q} \tag{1 a, b}$$

where V is the element volume, $\mathbf{T}$ a non-singular matrix easily evaluated on the basis of the vertex coordinates of the tetrahedrical finite element. An analogous comment holds for case (*c*). Therefore, the elastic plastic behavioural law described below for finite elements or structural constituents can be directly conceived as a material constitution. For other categories of ("refined") finite elements, formal procedures apt to generate element laws in generalized "natural" strains and stresses from elastic material laws have been pointed out in general by Corradi [8] and are only implicitly referred to here.

Keeping in mind the above preliminaries on its mechanical meaning and coverage, the following set of constitutive relations is adopted to describe the local deformability of constituents:

$$\mathbf{Q} = \mathbf{E}\boldsymbol{e} \tag{2}$$

$$\boldsymbol{\phi} = \tilde{\mathbf{N}}\mathbf{Q} - \mathbf{K} - \mathbf{R}(\boldsymbol{\lambda}) \leqq \mathbf{0} \tag{3 a, b}$$

$$\dot{\boldsymbol{\lambda}} \geqq \mathbf{0} \quad , \quad \tilde{\boldsymbol{\phi}}\dot{\boldsymbol{\lambda}} = 0 \tag{4 a, b}$$

$$\dot{\boldsymbol{p}} = \mathbf{N}\dot{\boldsymbol{\lambda}} \, . \tag{5}$$

Eq. (2) specifies the elastic behaviour, $\mathbf{E}$ being a symmetric positive definite matrix of elastic moduli. Eq. (3)-(5) govern the dissipative, non-holonomic (path-dependent, irreversible) plastic behaviour. Vector $\mathbf{K}$ collects positive constants or "yield limits". The yield functions are gathered in vector $\boldsymbol{\phi}$ and defined by (3*a*): by taking $\mathbf{N}$ as a constant matrix whose columns are unit vectors, they are assumed to be linear in the stresses, so that the instantaneous polyhedrical elastic domain $\boldsymbol{\phi} \leqq \mathbf{0}$ in the $\mathbf{Q}$ space, changes at yielding according to the "hardening rule" $\mathbf{R}(\boldsymbol{\lambda})$ assumed to be expressed by differentiable functions. This "rule" establishes a generally nonlinear dependence of the changes $\mathbf{R}$ of yield limits on the "internal variables" or "plastic multipliers" contained in vector λ. These variables are non-decreasing (4*a*), non-negative functions of time (as we will assume $\lambda = \mathbf{0}$ at $t = 0$) and play a key role in the present context. Each one represents a measure of the total amount of plastic flow, i.e. the contribution to plastic strains, due to the "activation" of the relevant yield plane; it can be also conceived as a measure of the sliding in a single direction of one of the one-dimensional dissipative slip devices contained in the element as a phenomenological interpretation of irreversible re-arrangements occurring at the microscale, see e.g. Martin [9], [10]. Therefore we prefer to call $\boldsymbol{\lambda}$ vector of "internal variables", instead of plastic multipliers as in several previous papers.

The "complementarity" equation (4 *b*), which applies componentwise in view of the sign constraints on the two vectors, rules out activation of a yield plane, say the i-th, if it does not contain the stress point $\mathbf{Q}$ ($\phi_i < 0$). Finally, eq. (5) expresses the associative nature of the plastic flow with respect to the yield surface ($\dot{\mathbf{p}}$ directed as the generalized outward normal to this surface).

The following remarks on the constitutive relation set (2)-(5) are of interest in what follows:

(*a*) As a consequence of (3*b*) and (4) one can write (" Prager's consistency rule "):

$$\tilde{\boldsymbol{\phi}}\dot{\boldsymbol{\lambda}} = 0 . \tag{6}$$

In fact, if $\dot{\lambda}_i > 0$ at instant t, then $\boldsymbol{\phi}_i = 0$ and $\boldsymbol{\phi}_i + \dot{\boldsymbol{\phi}}_i \, dt = 0$ by inequality (3*b*) and, hence, $\dot{\phi}_i = 0$; if $\dot{\phi}_i < 0$ and $\phi_i < 0$, then $\dot{\lambda}_i = 0$ by (4*b*); if $\dot{\phi}_i < 0$ and $\phi_i = 0$, then $\dot{\lambda}_i = 0$ again by (4*b*), which otherwise would be violated at $t + \delta t$.

(*b*) The plastic work, i.e. the energy dissipated in the whole element over the interval $0 \mapsto t$, using eqs. (5) and (3*a*) can be expressed in the forms:

$$\mathrm{D}(t) \equiv \int_0^t \mathbf{Q}(\tau)\, \dot{\mathbf{p}}(\tau)\, d\tau = \int_0^t \tilde{\dot{\lambda}} \tilde{\mathbf{N}} \mathbf{Q}\, d\tau = \int_0^t \tilde{\dot{\lambda}} (\boldsymbol{\phi} + \mathbf{K} + \mathbf{R})\, d\tau \tag{7}$$

or, through eq. (4 *b*):

$$\mathrm{D}(t) = \mathrm{D}'(t) + \mathrm{D}''(t) \tag{8}$$

having set:

$$\mathrm{D}'(t) \equiv \tilde{\mathbf{K}} \lambda(t) = \mathrm{D}'(\lambda) \tag{9}$$

$$\mathrm{D}''(t) \equiv \int_0^t \tilde{\mathbf{R}}[\lambda(\tau)]\, \dot{\lambda}(\tau)\, d\tau = \int_0^\lambda \tilde{\mathbf{R}}[\lambda(\tau)]\, d\lambda . \tag{10}$$

(*c*) Of the two addends into which the dissipated energy has been split, the former D′ can be called " perfect plasticity dissipation " as it is related to the original yield limits only; it is a linear function of the final internal variables $\lambda(t)$ and, hence, does not depend on the path $\lambda(\tau)$, $0 \leqq \tau \leqq t$, in the λ space. The latter addend D″ due to hardening (and referred to henceforth as " hardening dissipation ") generally depends on the path $\lambda(\tau)$ over $0 \leqq \tau \leqq t$. It does not and, hence, it becomes a function $\mathrm{D}''(\lambda)$ of the final distribution of internal variables $\lambda(t)$, if and only if the following alternative equivalent conditions hold (indices i and j running over all yield planes):

$$\mathrm{R}_i(\lambda) = \frac{\partial \mathrm{D}''(\lambda)}{\partial \lambda_i} \quad ; \quad \frac{\partial \mathrm{R}_i}{\partial \lambda_j} = \frac{\partial \mathrm{R}_j}{\partial \lambda_i} \tag{11 a, b}$$

$$\mathbf{H} \equiv \frac{\partial \tilde{\mathbf{R}}}{\partial \lambda} = \left[\frac{\partial^2 \mathrm{D}''}{\partial \lambda_i \, \partial \lambda_j}\right] = \frac{\partial \mathbf{R}}{\partial \lambda} = \tilde{\mathbf{H}} \, . \tag{11 c}$$

The equivalent conditions (11) for the existence of function $\mathrm{D}''(\lambda)$ require the symmetry of the "instantaneous hardening" matrix $\mathbf{H}(\lambda)$ (the Hessian matrix of D), i.e., mechanically speaking, a "reciprocal interaction" among yield modes at yielding.

(*c*) Assume that eqs. (11) holds, i.e. that there exists a hardening dissipated energy function $\mathrm{D}''(\lambda)$ and, hence, a total dissipation $\mathrm{D}(\lambda)$. This is convex, and so is $\mathrm{D}(\lambda)$, if and only if:

$$\mathrm{D}''(\lambda) \geqq \mathrm{D}''(\lambda^*) + \tilde{\mathbf{R}}(\lambda^*)(\lambda - \lambda^*) \, , \qquad \forall \, \lambda \, , \lambda^* \, . \tag{12}$$

In fact, $\mathbf{R}(\lambda^*)$ is the gradient of $\mathrm{D}''(\lambda)$ in λ^*, eq. (11*a*).

(*d*) For linear hardening matrix $\mathbf{H}$ is constant. Then its symmetry implies that D'' is a quadratic form of λ, its positive semi-definiteness that D'' is convex. Under this restriction one goes back to the piecewise linear constitutive model assumed for the theory developed in [6], [7] and for subsequent developments by various Authors.

3. Basic relations for discrete structural models

An aggregate of the elements described in Sec. 2 will be considered here with the following additional nomenclature and conventions. The relations (2)-(5) governing the local, element deformability will be conceived henceforth as covering all m members or elements simultaneously, e.g.: $\mathbf{Q} \equiv \{\tilde{\mathbf{Q}}^1 \ldots \tilde{\mathbf{Q}}^m\}$; $\tilde{\boldsymbol{\phi}} \equiv \{\tilde{\boldsymbol{\phi}} \ldots \tilde{\boldsymbol{\phi}}^m\}$; $\mathbf{E} \equiv \mathrm{diag}\,[\mathbf{E}^1 \ldots \mathbf{E}^m]$; $\mathbf{N} \equiv \mathrm{diag}\,[\mathbf{N}^1 \ldots \mathbf{N}^m]$ etc. Subscript o denotes quantities pertaining to the initial static situation Σ_0 supposed to be elastic ($\lambda_0 = \mathbf{0}$) and strictly stable (with positive definite stiffness matrix); subscript ∞ will mark their asymptotic values in time (at $t = \infty$). Let the variable repeated external actions be equivalent nodal forces collected in n-vector $\mathbf{F}(t)$ and imposed (e.g. thermal) generalized strains defined by vector $\boldsymbol{\theta}(t)$. The n-vector $\boldsymbol{u}(t)$ and the vectors $\mathbf{Q}(t)$, $\boldsymbol{p}(t)$ etc. define the unknown quasi-static response of the system to the above loads in terms of nodal displacements, stresses, plastic strains etc., respectively. They are governed by the constitutive eqs. (2)-(5) encompassing all elements and, under the hypothesis of "small deformations", by the linearized equations of compatibility and equilibrium:

$$\mathbf{C}\boldsymbol{u} = \boldsymbol{e} + \boldsymbol{p} + \boldsymbol{\theta} \equiv \boldsymbol{q} \tag{13}$$

$$\tilde{\mathbf{C}}\mathbf{Q} + \mathbf{G}(\mathbf{Q}_0)\,\boldsymbol{u} = \mathbf{F}\,. \tag{14}$$

Here matrix $\mathbf{C}$ depends only on the geometry of $\bar{\Sigma}$; the (symmetric) "geometric stiffness" matrix $\mathbf{G}$ depends linearly on $\mathbf{Q}_0$; both are constant in time. The pre-existing stresses $\mathbf{Q}_0$ intervene also in (3) but can conveniently be accomodated in the original yield limit vector, which becomes:

$$\mathbf{K} = \mathbf{K}_0 - \tilde{\mathbf{N}}\mathbf{Q}_0 \tag{15}$$

$\mathbf{K}_0$ being the original yield limit vector in an hypothetical stressless state.

For the above defined structural model, linear-elastic displacement and stress responses to the external actions can be expressed as follows:

$$\boldsymbol{u}^e = \mathbf{S}^{-1}\,\mathbf{F} + \mathbf{S}^{-1}\,\tilde{\mathbf{C}}\mathbf{E}\boldsymbol{\theta}\ ; \qquad \text{where:}\quad \mathbf{S} \equiv \tilde{\mathbf{C}}\mathbf{E}\mathbf{C} + \mathbf{G} \tag{16 a, b}$$

$$\mathbf{Q}^e = \mathbf{E}\mathbf{C}\mathbf{S}^{-1}\,\mathbf{F} + \mathbf{Z}\boldsymbol{\theta}\ ; \qquad \text{where:}\quad \mathbf{Z} \equiv \mathbf{E}\mathbf{C}\mathbf{S}^{-1}\,\tilde{\mathbf{C}}\mathbf{E} - \mathbf{E} \tag{17 a, b}$$

It is assumed here that the elastic equilibrium is stable in the starting situation Σ_0, i.e. that the overall elastic stiffness matrix $\mathbf{S}$, eq. (16*b*), is positive definite. Displacements and stresses due to plastic strains $\boldsymbol{p}$ are obtained from (11) and (17), respectively, for $\mathbf{F} = \mathbf{0}$, $\boldsymbol{\theta} = \boldsymbol{p} = \mathbf{N}\lambda$:

$$\boldsymbol{u}^p = \mathbf{S}^{-1}\,\tilde{\mathbf{C}}\mathbf{E}\mathbf{N}\boldsymbol{\lambda} \tag{18}$$

$$\mathbf{Q}^p = \mathbf{Z}\mathbf{N}\boldsymbol{\lambda}\,. \tag{19}$$

4. GENERALIZED FIRST SHAKEDOWN THEOREMS

On the basis of the preliminaries of Secs. 2 and 3 and in view of subsequent use, we define the following energy quantity for the whole system:

$$\mathrm{U} \equiv \mathrm{D}''(\boldsymbol{\lambda}) + \frac{1}{2}\tilde{\boldsymbol{\lambda}}\mathbf{W}\boldsymbol{\lambda}\,, \tag{20}$$

having set:

$$\mathrm{D}'' \equiv \sum_h^m \mathrm{D}_h''(\lambda_h) \quad ; \quad \mathbf{W} \equiv -\tilde{\mathbf{N}}\mathbf{Z}\tilde{\mathbf{N}} = \tilde{\mathbf{W}}\,. \tag{21 a, b}$$

Clearly, in general U is a functional of the whole yielding process $\dot{\lambda}(\tau)$, $0 \leq \tau \leq t$, i.e. a path-dependent function of $\boldsymbol{\lambda}$, as the hardening dissipated energy D_h'' is so; U becomes function $\mathrm{U}(\lambda)$ of the distribution of internal variables λ at time t, if D_h is so for all m elements, i.e. if all constituents exhibit (reciprocal) hardening complying with eqs. (11).

With reference to discrete or discretized structures governed by eqs. (2)-(5) and (13)-(15), the following sufficient criterion for adaptation will be proved below.

THEOREM 1. *When the hardening behaviour exhibits reciprocal interaction, eq.* (11), *and the energy function* U ($\boldsymbol{\lambda}$) *is convex, then shakedown occurs if a constant internal variable vector* $\boldsymbol{\lambda}^*$ *exists such that:*

$$\phi^* \equiv \tilde{\mathbf{N}}\mathbf{Q}^e - \mathbf{W}\boldsymbol{\lambda}^* - \mathbf{K} - \mathbf{R}(\boldsymbol{\lambda}^*) < \mathbf{0}\,, \qquad \forall t \geqq 0 \tag{21}$$

Proof. For the actual evolution of the structure the stress distribution at any $t \geqq 0$ is conceived as the superposition of elastic stress responses to the current external actions (17) and to the plastic strains (19) developed up to t (Colonnetti's interpretation of structural plasticity):

$$\mathbf{Q} = \mathbf{Q}^e + \mathbf{Z}\boldsymbol{p}\,. \tag{22}$$

Account taken of eq. (22) and of the constitutive relations (3), (5) interpreted as concerning all elements (see Sec. 3), we can write for the current yield functions:

$$\phi = \mathbf{N}\mathbf{Q}^e - \mathbf{W}\lambda - \mathbf{K} - \mathbf{R}(\lambda) \leqq \mathbf{0}\,, \qquad \forall t \geqq 0\,. \tag{23}$$

Let us define the time function:

$$\mathrm{L}(t) \equiv \mathrm{U}(\lambda) - \mathrm{U}(\lambda^*) - (\tilde{\lambda} - \tilde{\lambda}^*)\left(\frac{\partial \mathrm{U}}{\partial \boldsymbol{\lambda}}\right)_{\lambda^*}. \tag{24}$$

It will be called Ljapunov function in view of similarities to well known procedures in dynamic stability problems. The following circumstances are worth noting.

(*a*) Because of the hypothesis of reciprocal hardening in the sense of eqs. (11), the hardening dissipated energy D″ is a (path-independent) function of the internal variable vector. Therefore, function U (λ) according to (20) exists and, hence, so does function L (t) for any cho en λ^*, whatever the actual plastic evolution λ (t) of the system may be.

(*b*) Because of the convexity hypothesis, function L (t) is non-negative: L (t) $\geqq$ 0 for any t.

Through eqs. (10) and (20) and easy manipulations, the Ljapunov function can be given the more explicit expression:

$$\text{(25)}\quad \mathrm{L}(t) = \mathrm{D}''(\lambda) - \mathrm{D}''(\lambda^*) - \tilde{\mathbf{R}}(\lambda^*)(\lambda - \lambda^*) + \frac{1}{2}(\tilde{\lambda} - \tilde{\lambda}^*)\,\mathbf{W}\,(\lambda - \lambda^*)\,.$$

Its time derivative reads, by virtue of (10):

$$\text{(26)}\qquad \dot{\mathrm{L}}(t) = \tilde{\mathbf{R}}(\lambda)\,\dot{\lambda} - \tilde{\mathbf{R}}(\lambda^*)\,\dot{\lambda} + (\tilde{\lambda} - \tilde{\lambda}^*)\,\mathbf{W}\,\dot{\lambda}$$

or, alternatively, using (10), (21), (22) and constitutive relations and marking by an asterisk quantities related to the fictitious constant plastic strains $\boldsymbol{p}^* = \mathbf{N}\lambda^*$ through (22):

$$\begin{aligned} \dot{\mathrm{L}}(t) &= \tilde{\dot{\lambda}}\,[\mathbf{R}(\lambda) - \mathbf{R}(\lambda^*) - \tilde{\mathbf{N}}\mathbf{Q}^p + \tilde{\mathbf{N}}\mathbf{Q}^{p*}] = \\ \text{(27)}\qquad &= \tilde{\dot{\lambda}}\,[\tilde{\mathbf{N}}\mathbf{Q}^{p*} + \tilde{\mathbf{N}}\mathbf{Q}^e - \mathbf{R}(\lambda^*) - \mathbf{K} - ((\tilde{\mathbf{N}}\mathbf{Q}^p + \tilde{\mathbf{N}}\mathbf{Q}^e - \mathbf{R}(\lambda) - \mathbf{K})] \\ \dot{\mathrm{L}}(t) &= \tilde{\dot{\boldsymbol{\lambda}}}\boldsymbol{\phi}^* - \tilde{\dot{\boldsymbol{\lambda}}}\boldsymbol{\phi} = \tilde{\dot{\boldsymbol{\lambda}}}\boldsymbol{\phi}^*\,. \end{aligned}$$

The hypothesis inequality (21), together with (27), implies that:

$$\text{(28)}\qquad \dot{\mathrm{L}}(t) \leqq 0\,; \qquad \dot{\mathrm{L}}(t) = 0 \qquad \text{if and only if} \quad \dot{\lambda} = \mathbf{0}\,.$$

The hypothesis (21) means also that there is a number $\beta > 1$ such that:

$$\text{(29)}\qquad \tilde{\mathbf{N}}\mathbf{Q}^e - \mathbf{W}\lambda^* - \mathbf{R}(\lambda^*) \leqq \beta^{-1}\,\mathbf{K}\,, \qquad \forall t \geqq 0$$

whence, pre-multiplying by $-\dot{\lambda} \leqq \mathbf{0}$:

$$\text{(30)}\qquad -\tilde{\dot{\lambda}}\tilde{\mathbf{N}}\mathbf{Q}^e - \tilde{\dot{\lambda}}\mathbf{W}\lambda^* + \tilde{\dot{\lambda}}\mathbf{R}(\lambda^*) \geqq -\beta^{-1}\,\tilde{\mathbf{K}}\dot{\lambda}\,.$$

By summing (27) and (28) and taking into account (26), we derive a crucial link between the Ljapunov function L (t) and the actual time evolution of the system:

$$\text{(31)}\qquad -\dot{\mathrm{L}}(t) \geqq (1 - \beta^{-1})\,\tilde{\mathbf{K}}\dot{\lambda}\,, \qquad \forall t \geqq 0$$

By integration in time from $t = 0$ to $t = \infty$:

$$\text{(32)}\qquad \mathrm{L}(0) - \mathrm{L}(\infty) \geqq (1 - \beta^{-1})\,\tilde{\mathbf{K}}\lambda_\infty\,.$$

Since L has been shown to be a non-negative and non-increasing function of time under the hypothesis stated, this inequality assures that λ_∞ is bounded, i.e. that the system will shakedown under the history of external action $\mathbf{F}(t)$, $\boldsymbol{\theta}(t)$.

THEOREM 2. *Shakedown cannot occur if no constant internal variable vector λ^* exists such that:*

$$\tilde{\mathbf{N}}\mathbf{Q}^e - \mathbf{W}\lambda^* - \mathbf{K} - \mathbf{R}(\lambda^*) \leqq \mathbf{0}\,, \qquad \forall t \geqq 0\,. \tag{33}$$

Proof. By its very definition, shakedown means that, eventually in time, the actual plastic strain distribution will be defined by a bounded vector λ_∞ and no further yielding will occur. Hence, inequalities (31) are satisfied for $\lambda^* = \lambda_\infty$ after adaptation in an unlimited time interval. In such interval the parameters governing the external actions attain all values included in the assigned loading domain, this fact being implicit in the very notion of (quasi-static) variable repeated loads. Therefore, after shakedown they will reach any values assumed before it and, hence, if adaptation occurs, inequality (31) must be fulfilled for all $t \geqq 0$ at least by $\lambda^* = \lambda_\infty$. If it cannot be fulfilled by any λ^* for all $t \geqq 0$, then adaptation is ruled out.

It is worth noting that the above statement, in contrast to the former, is not subjected to any restriction on the hardening (or softening), nor to any condition on the energy U (t).

5. DETERMINATION OF THE SAFETY FACTOR

As in the classical context, through a customary argument founded on a perturbation $\delta\mathbf{K} > \mathbf{0}$ on the original yield limits $\mathbf{K}$, it is possible and useful for numerical applications to combine the two statements proved in Sec. 5 into a single statement cf. e.g. [1], [9]. This provides a necessary and sufficient criterion for shakedown in the following terms:

Unified statement. When the hardening behaviour (of all elements) exhibits reciprocal interaction, eq. (11), and the energy function U (λ) is convex, then shakedown occurs if and only if a constant internal variable vector λ^* exists such that:

$$\tilde{\mathbf{N}}\mathbf{Q}^e(t) - \mathbf{W}\lambda^* - \mathbf{R}(\lambda^*) \leqq \mathbf{K}\,, \qquad \forall t \geqq 0\,. \tag{34}$$

In order to confer an operative, computational interest to the above criterion, consider the projection of the elastic stress vector $\mathbf{Q}^e_h(t)$ in element h on the outward normal unit vector $\mathbf{N}^h_j$ relative to the j-th yield plane in the stress space of that element. Evaluate its maximum over the given "loading domain" in the space of the parameters governing the external actions $\mathbf{F}(t)$, $\theta(t)$. Then, for all j and $h = 1 \dots m$, generate the vector:

$$\mathbf{M} = \{\dots \mathrm{M}^h_j \dots\}\,, \qquad \text{with} \quad \mathrm{M}^h_i = \max_t \{\tilde{\mathbf{N}}^h_i \mathbf{Q}^e_h(t)\}\,. \tag{35 a, b}$$

Consider a common multiplier of all external actions or " load factor " $\alpha \geqq 0$. In the jargon of engineering plasticity, " safety factor " with respect to inadaptation (and, hence, eventual failure) is the value s of α such that for $\alpha \leqq s$ the structure shakes down, for $\alpha > s$ it does not.

The above unified statement clearly permits to cast the search for the safety factor into the following non-linear programming problem:

$$s = \max_{\alpha, \boldsymbol{\lambda}^*} \alpha , \qquad \text{subject to: } \mathbf{M}\alpha - \mathbf{W}\boldsymbol{\lambda}^* - \mathbf{R}(\boldsymbol{\lambda}^*) \leqq \mathbf{K} . \tag{36}$$

Qualitative and computational features (such as conditions for convexity and solvability, dualization, etc.) of this problem will be discussed elsewhere. We anticipate here the remark that in (36) the feasible region (and, hence, the programming problem itself) is convex if the components of vector $\mathbf{R}$ are concave functions of λ.

6. MECHANICAL INTERPRETATIONS AND SPECIALIZATIONS TO PREVIOUS RESULTS

The physical meaning and implications of the convexity condition to which the results of Sec. 4 are subjected, can be elucidated as follows.

THEOREM 3. *The convexity of the energy function* U (λ), *eq.* (20), *entails the overall stability of the system, whatever set of yielding modes may become active in its evolution.*

Proof. Overall stability here means that the second order work required by any infinitesimal configuration changes and performed by the external force changes needed to preserve equilibrium cannot be negative:

$$\frac{1}{2} \tilde{\dot{\mathbf{F}}} \dot{\boldsymbol{u}} \delta t^2 \geqq 0 , \qquad \forall \dot{\boldsymbol{u}} . \tag{39}$$

Note first that, by virtue of (16), (18):

$$\tilde{\dot{\mathbf{F}}} \dot{\boldsymbol{u}} = \tilde{\dot{\mathbf{F}}} (\dot{\boldsymbol{u}}^e + \dot{\boldsymbol{u}}^p) = \tilde{\dot{\boldsymbol{u}}}^e \mathbf{S} \dot{\boldsymbol{u}}^e + \tilde{\dot{\mathbf{F}}} \dot{\boldsymbol{u}}^p . \tag{40}$$

For any set of compatible (but otherwise arbitrary) kinematic variables (primed) and any set of quantities representing an actual statical process, we can write:

$$\tilde{\mathbf{F}} \boldsymbol{u}' = \tilde{\mathbf{Q}} \boldsymbol{q}' + \tilde{\boldsymbol{u}} \mathbf{G} \boldsymbol{u}' . \tag{41}$$

This is readily verified using (13), (14) and reduces to an usual virtual wokr equation for $\mathbf{G} = \mathbf{0}$. Applying eq. (39) twice, one obtains:

$$\tilde{\mathbf{F}}\dot{\boldsymbol{u}}^p = \tilde{\dot{\mathbf{Q}}}^e(\dot{\boldsymbol{e}}^p + \dot{\boldsymbol{p}}) + \tilde{\dot{\boldsymbol{u}}}^e\mathbf{G}\dot{\boldsymbol{u}}^p \quad ; \quad \tilde{\dot{\mathbf{Q}}}^p\dot{\boldsymbol{e}}^e + \tilde{\dot{\boldsymbol{u}}}^p\mathbf{G}\dot{\boldsymbol{u}}^e = 0 \tag{42 a, b}$$

whence, through the symmetry of matrices **G** and **E**:

$$\tilde{\mathbf{F}}\dot{\boldsymbol{u}}^p = \tilde{\dot{\mathbf{Q}}}^e\dot{\boldsymbol{p}} + \tilde{\dot{\mathbf{Q}}}^e\dot{\boldsymbol{e}}^p - \tilde{\dot{\mathbf{Q}}}^p\dot{\boldsymbol{e}}^e = \tilde{\dot{\mathbf{Q}}}^e\dot{\boldsymbol{p}}\,. \tag{43}$$

Now, taking the time derivative of the yield functions (23)

$$\dot{\boldsymbol{\phi}} = \tilde{\mathbf{N}}\dot{\mathbf{Q}}^e - \mathbf{W}\dot{\boldsymbol{\lambda}} - \mathbf{H}(\boldsymbol{\lambda})\dot{\boldsymbol{\lambda}} \tag{44}$$

and using the constitutive equations (5), (6), we obtain:

$$\tilde{\dot{\mathbf{Q}}}^e\dot{\boldsymbol{p}} = \tilde{\dot{\boldsymbol{\lambda}}}(\mathbf{W} + \mathbf{H})\dot{\boldsymbol{\lambda}}\,. \tag{45}$$

Substitutions of (43) into (41) and of this into (38) lead to:

$$\tilde{\mathbf{F}}\dot{\boldsymbol{u}} = \tilde{\dot{\boldsymbol{u}}}^e\mathbf{S}\dot{\boldsymbol{u}}^e + \tilde{\dot{\boldsymbol{\lambda}}}(\mathbf{W} + \mathbf{H})\dot{\boldsymbol{\lambda}}\,. \tag{46}$$

The former addend on the r.h.s. of (44) is non-negative because of the assumed stability of the (elastic) starting situation ($\mathbf{F} = \mathbf{0}$, $\boldsymbol{\lambda} = \mathbf{0}$ at $t = 0$); the matrix in the latter is the Hessian matrix of the energy function U and, hence, is positive semi-definitive if U is convex. Thus stability in the sense (35) is guaranteed under the stated hypotheses.

It is worth noting in passing that the convexity condition above shown to be sufficient for overall stability, is by no means necessary for it.

Also noteworthy is the trade-off between matrix **W** (containing the elastic and connectivity properties and, through **G**, the geometric effects) and the hardening matrix **H** concerning local plastic behaviour. In principle, lack of convexity of one addend can be compensated by the other.

If the configuration changes are assumed to have no influence on equilibrium ($\mathbf{G} = \mathbf{0}$; "first order", small deformations theory) matrix **W** is always positive semi-definite and, hence, the relevant addend in the expression of the energy function U is certainly convex. Then possible instabilizing effects jeopardizing the convexity of U$(\boldsymbol{\lambda})$ may only be provoked by material instability ("softening") reflected by the Hessian hardening matrix $\mathbf{H}(\boldsymbol{\lambda})$.

Particularizations to earlier results by adopting more restrictive assumptions may provide further insight into and a clearer perspective for the results presented in what precedes.

For linear hardening behaviour ($\mathbf{H}$ = constant, i.e. **R** depends linearly on $\boldsymbol{\lambda}$), the results of ref. [7] are recovered. If configuration changes are assumed to hav eno influence on equilibrium ($\mathbf{G} = \mathbf{0}$), then $\mathbf{Q}^p = \mathbf{Z}\boldsymbol{p}$ are selfequilibrated stresses.

For non-hardening systems ($\mathbf{R} = \mathbf{0}$, $\mathbf{H} = \mathbf{0}$) constant selfstresses (or redundant internal forces which govern them) can be assumed as test variables in the shakedown criterion, instead of the internal variables λ^*. Then, the conditions on the energy function $U(\lambda)$ are certainly trivially satisfied and the classical Melan's theorem for the present classes of discrete structures, is arrived at (see e.g. [9], [11], [12]).

7. CLOSING REMARKS

In what precedes "a priori" shakedown conditions have been established and interpreted mechanically for elastic plastic structures with local behaviour which exhibits generally non-linear hardening. "A priori" means applicable without performing the evolutive non-linear analysis, i.e. resting on constitutive properties and on linear elastic responses to external actions. Thus previous works have been further extended and generalized.

The following limitations should be noticed. As for the material (or local constituent) behaviour law, the yield functions have been linearized with respect to (generalized) stresses and for each yield mode a non-decreasing internal variable has been assumed to measure its contribution to plastic (generalized) strains. This assumption is uncontroversial for one-component structures such as trusses, beams and frames, but represents only a possible convenient approximation of the constitution usually adopted for multicomponent structures and continua. The effects of geometry changes have been allowed for here only through an additional term linear in the displacements and in the pre-existing stresses (at $t = 0$, stresses due to pre-existing constant loads). This assumption may be unrealistic in many situations.

In the presence of the geometric and/or physical (hardening) stabilizing effects, the safety factor with respect to inadaptation is expected to be very sensitive to the trade-off between the two effects and, in particular, to the hardening law assumed. Therefore, its numerical determination "per se" may be of limited practical interest (e.g. with isotropic hardening shakedown always occurs for obvious reasons). However, the present results concerning shakedown conditions of Melan's type are intended as a basis for a broader generalized theory (to be presented elsewhere), including deformation bounding theorems and techniques and their extensions to dynamics.

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