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A non-local theory of superfluidity

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Meccanica dei fluidi. — *A non-local theory of superfluidity* (*). Nota di MAURO FABRIZIO e GIORGIO GENTILI (**), presentata (***) dal Corrisp. T. MANACORDA.

ABSTRACT. — We will formulate a macroscopic theory of Superfluidity, using a particular constitutive equation of differential form which we will demonstrate to be equivalent to a non-local relation between the stress and the density.

KEY WORDS: Superfluidity; Constitutive; Equations.

RIASSUNTO. — *Una teoria non-locale della superfluidità.* Viene formulata una teoria macroscopica per la Superfluidità, facendo uso di una particolare equazione costitutiva di tipo differenziale che si dimostra essere equivalente ad una relazione non locale tra gli sforzi e la densità.

1. INTRODUCTION

As in macroscopic Landau theory we suppose HeII fluid composed of a mixture of *normal fluid* and *superfluid*, whose density ρ_n, ρ_s and velocity $\boldsymbol{v}_n$, $\boldsymbol{v}_s$ are related by:

$$\rho = \rho_n + \rho_s \quad , \quad \rho \boldsymbol{v} = \rho_n \boldsymbol{v}_n + \rho_s \boldsymbol{v}_s \tag{1}$$

It is well-known that the ratio ρ_n/ρ goes to zero when the temperature goes to absolute zero, i.e. HeII is all in superfluid state at zero absolute temperature. Therefore $\rho = \rho_s$, $\boldsymbol{v} = \boldsymbol{v}_s$ and the motion equations will be (see Putterman [1], Atkin and Fox [2-3]):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} = 0 \tag{2}$$

$$\rho \dot{\boldsymbol{v}} = - \rho \nabla \mu + \rho \boldsymbol{b} \tag{3}$$

where the vector $\boldsymbol{f} = \nabla \mu$ represents the action exerted by internal forces, $\boldsymbol{b}$ the body forces. Because the superfluid component is able to flow without

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resistence, the work of internal forces is necessarily equal to zero on every cyclic process; i.e. exists a function e, called *internal energy*, such that:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} e \, \mathrm{d}m = \int_{\Omega} \rho \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x \tag{4}$$

where $\Omega \subset \mathbf{R}^3$ is the domain in which the superfluid is placed.

Now we have to explain the constitutive equation for the vector $\boldsymbol{f} = \nabla\mu$.

In Landau theory [4] the dissipationless behaviour is explained by means of a constitutive equation for μ similar to the perfect fluid one, i.e. he supposed the Pascal's law $\mu = \mu(\rho)$. Furthermore, on the grounds of microscopic considerations, Landau hypothesizes the condition

$$\nabla \times \boldsymbol{v}_s = 0 \tag{5}$$

that has the meaning of a kinematic constraint.

Such an approach seems questionable because it doesn't take into account the peculiar properties of superfluids, different from the perfect ones. Furthermore condition (5) cannot be thought of as characteristic of superfluids only, because when the fluid is irrotational initially, it can be obtained by (3) through considerations valid also for perfect fluids. In other words according to the Landau theory it should always be possible to obtain a superfluid from a perfect one when the initial conditions are suitably chosen.

Moreover relation (5) can be also questioned because it is invariant for changes in the frame of reference [2-3] as should happen for every equation representing a constraint.

Finally, as Putterman writes in [1], " the central macroscopic problem is that of finding the modification of two fluid thermo-hydrodynamics to more fully include the macroscopic quantum effects . . . for these effects the stresses are not isotropic depending on the deformation, so that Pascal's Law is not obtained ". Namely the mascroscopic quantum effects point out a non-local constitutive equation between stress and density.

Following the idea proposed by London [5] for the study of superconductivity, we will make use of a constitutive equation of differential type, which we will demonstrate to be equivalent to a non-local equation [6], as happens in superconductivity. Such a constitutive equation allows us to explain the absence of viscosity and leads to a non-kinematic constraint which generalizes (5).

2. A non-Local constitutive equation for HeII

We suppose that HeII is all in superfluid state, i.e. $\rho_n = 0$ and the motion equations are given by (2), (3).

*CONSTITUTIVE HYPOTESIS. For a superfluid material we will suppose the vector **f** related to the density* ρ *through the differential equation:*

$$\nabla \cdot \boldsymbol{f} = \lambda \rho \tag{6}$$

where λ *is a suitable parameter characteristic of the material.*

We will discuss now the consequence of equation (6). If we derive with respect to time and take into account equation (2), we have:

$$\nabla \cdot \frac{\partial \boldsymbol{f}}{\partial t} = -\lambda \nabla \cdot (\rho \boldsymbol{v}) \tag{7}$$

from which we obtain

$$\frac{\partial \boldsymbol{f}}{\partial t} = -\lambda \rho \boldsymbol{v} + \nabla \times \mathbf{A} \tag{8}$$

where $\mathbf{A}$ is an arbitrary vector function. Furthermore, from (7) integrating on the domain Ω, we have:

$$\int_{\partial\Omega} \frac{\partial \boldsymbol{f}}{\partial t} \cdot \boldsymbol{n} \, d\sigma = -\int_{\partial\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma = 0 \tag{9}$$

It is therefore compatible with this last relation the requirement that on the part of the boundary of Ω resulting a rigid wall we have:

$$\boldsymbol{f} \cdot \boldsymbol{n} = 0 \qquad \text{on} \quad \partial\Omega . \tag{10}$$

As a consequence $\boldsymbol{f} = \nabla \mu$ results parallel to the surface where the fluid is contained. Such condition accounts for the so called "fountain effect".

In fact in order for the fluid to be able to climb on the wall it is necessary for the gravity to be overcome by the internal force $\boldsymbol{f}$ along the surface of the containing vessel. Moreover making use of (8) we can prove the absence of dissipation. Let us consider in fact the integral

$$-\int_{\Omega} \boldsymbol{f} \cdot \rho \boldsymbol{v} \, dx = 1/\lambda \int_{\Omega} \{\boldsymbol{f} \cdot \partial \boldsymbol{f}/\partial t - \boldsymbol{f} \cdot \nabla x \mathbf{A}\} \, dx \tag{11}$$

where we have taken into account (8). Because of the divergence theorem we have

$$-\int_{\Omega} \boldsymbol{f} \cdot \rho \boldsymbol{v} \, dx = \frac{1}{2\lambda} \frac{\partial}{\partial t} \int_{\Omega} \boldsymbol{f}^2 \, dx - \frac{1}{\lambda} \int_{\partial\Omega} \mu \nabla \times \mathbf{A} \cdot \boldsymbol{n} \, d\sigma + \tag{12}$$
$$+ \frac{1}{\lambda} \int_{\Omega} \mu \nabla \cdot \nabla \times \mathbf{A} \, dx$$

Therefore, being $\nabla \cdot \nabla \times \mathbf{A} = 0$ on Ω and $\nabla \times \mathbf{A} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ it results:

$$\frac{1}{2\lambda} \frac{\partial}{\partial t} \int_\Omega \boldsymbol{f}^2 \, \mathrm{d}x = - \int_\Omega \boldsymbol{f} \cdot \rho \boldsymbol{v} \, \mathrm{d}x \tag{13}$$

The equality (4) is proved and then the absence of dissipation.

The constitutive equation (6) is given in differential form. It is however also possible to give it an expression as a functional equation. In fact equation (6) must be integrated with the boundary condition

$$\begin{aligned} \boldsymbol{f} \cdot \boldsymbol{n} &= 0 \quad \text{on} \quad \partial\Omega_1 \\ \mu &= \mu_0 \quad \text{on} \quad \partial\Omega_2 = \partial\Omega \setminus \partial\Omega_1 \end{aligned} \tag{14}$$

where $\partial\Omega_2$ is the part of the boundary that is free. The problem can then be written as:

$$\nabla \cdot \nabla\mu = \lambda\rho \quad \text{on} \quad \Omega \tag{15}$$

$$\nabla\mu \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_1 \tag{16}$$

$$\mu = \mu_0 \quad \text{on} \quad \partial\Omega_2 . \tag{17}$$

When we suppose the density ρ on Ω known, the problem becomes the classical problem of Dirichet-Newmann for the Poisson equation, that admits solution under wide hypothesis of regularity; moreover the solution is unique when $\partial\Omega_2 \neq \varnothing$. Under such regularity hypothesis it is possible to express μ as a function of $\rho(\cdot, t)$ through the functional:

$$\mu(x, t) = \mu_x(\rho(\cdot, t)) . \tag{18}$$

One can notice that the fluid is not barotropic; the constitutive equation turns out to be non-local, analogously with what one finds in superconductivity. Equation (18) has then the meaning of the constitutive equation for the superfluid, since it contains equation (6). Moreover the relation (18) is certainly objective, because it relates scalar quantities.

Finally from (8), since $\boldsymbol{f} = \nabla\mu$, one obtains:

$$\lambda\nabla \times \rho\boldsymbol{v} = \nabla \times \nabla \times \mathbf{A} \tag{19}$$

where $\mathbf{A}$ is such that

$$\mathbf{A} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_1 . \tag{20}$$

Among the possible choices for $\mathbf{A}$ there are the ones for which

$$\nabla \times \nabla \times \mathbf{A} = 0 \quad \text{on} \quad \Omega\,, \quad \nabla \times \mathbf{A} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_1\,.$$

In such a case from (19) one obtains

$$\nabla \times \rho \boldsymbol{v} = 0\,. \tag{21}$$

This condition, which is not exactly coincident with the Landau hypothesis (5), is not the only constraint for the momentum $\rho\boldsymbol{v}$ because such constraints are related by equation (19).

Now we consider the case in which the temperature θ is not zero but constant on all fluid. Then HeII is described by means of a mixture of two fluids non-interacting. The basic variables of the theory are the densities ρ_n, ρ_s which are related to the density ρ and velocity $\boldsymbol{v}$ by (1).

As in Landau theory [4], the continuity equations and the balances of linear momentum for the normal and superfluid components are expressed in conventional form:

$$\begin{aligned} &\frac{\partial \rho_n}{\partial t} + \nabla \cdot \rho_n \boldsymbol{v}_n = 0 \\ &\frac{\partial \rho_n \boldsymbol{v}_n}{\partial t} + \nabla \cdot (\rho_n \boldsymbol{v}_n \otimes \boldsymbol{v}_n) = \nabla \cdot \mathbf{T}_n + \rho_n \boldsymbol{b} \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \boldsymbol{v}_s) = 0 \\ &\frac{\partial \rho_s \boldsymbol{v}_s}{\partial t} + \nabla \cdot (\rho_s \boldsymbol{v}_s \otimes v_s) = \rho_s \nabla\mu + \rho_s \boldsymbol{b} \end{aligned} \tag{23}$$

with

$$\mathbf{T}_n = (-p + \lambda' \nabla \cdot \boldsymbol{v}_n)\, \mathrm{I} + 2\mu' \mathbf{D} \tag{24}$$

where λ', μ' are the usual coefficients of Navier-Stokes stress tensor. Instead of classical Pascal's law for μ, we assume the new constitutive equation

$$\nabla \cdot \nabla\mu = \lambda \rho_s\,.$$

For a mixture the Dissipation Principle (or Second Law of Thermodynamics for isothermal processes) has the form:

DISSIPATION PRINCIPLE.

For any cyclic process

$$\oint_0^T \int_\Omega \mathbf{T}_n \cdot \nabla \boldsymbol{v}_n \, dx \, dt + \oint_0^T \int_\Omega \rho_s \nabla\mu \cdot \boldsymbol{v}_s \, dx \, dt \geq 0 \tag{25}$$

where T *is the duration of cyclic process.*

The first integral is able to explain the dissipation effects that HeII show, the second integral accounts for the superfluid behaviour. In other words, necessarily we have by (24) that

$$\int_\Omega \mathbf{T}_n(t) \cdot \nabla \boldsymbol{v}_n(t) \, dx = \int_\Omega | \lambda' (\nabla \cdot \boldsymbol{v}_n(t))^2 + 2\mu' \mathbf{D}_n^2(t) | \, dx \geq 0$$

whereas by (12), (13) and (4) we obtain:

$$\oint_0^T \int_\Omega \rho_s \nabla\mu \cdot \boldsymbol{v}_s \, dx \, dt = 0 .$$

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