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On $2p$-dimensional Riemannian manifolds with positive scalar curvature

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Geometria. — *On 2p–dimensional Riemannian manifolds with positive scalar curvature.* (*) Nota (**) di DOMENICO PERRONE (***), presentata dal Corrisp. E. VESENTINI.

RIASSUNTO. — In questo lavoro si danno alcuni risultati sugli spettri degli operatori di Laplace per varietà Riemanniane compatte con curvatura scalare positiva e di dimensione $2p$. Ad essi si aggiunge una osservazione riguardante la congettura di Yamabe.

1. INTRODUCTION

Let (M, g) be a n–dimensional compact Riemannian manifold with scalar curvature τ and Ricci tensor ρ. In this paper all manifolds are assumed to be orientable, connected and C^∞. By (S^n, g_0) we denote the n–sphere with the canonical metric of constant sectional curvature 1 and by $\wedge^2(M)$ the space of the 2–forms on M.

If (M, g) is conformally flat, the following results are known.

A) If τ is a constant and ρ is positive definite, then (M, g) is a space of constant sectional curvature (cf. [6]).

B) If τ is constant and the length of ρ is less than $\tau / \sqrt{n-1}$, $n \geq 3$, then (M, g) is a space of constant sectional curvature (cf. [2]).

C) If $\rho \geq (n-1) g$, then the minimal eigenvalue ${}^q\lambda_1$ of the Laplace operator for q–forms satisfies: ${}^q\lambda_1 \geq q(n-q+1)$ for $1 \leq q \leq n/2$; moreover, if M is simply connected, equality holds if, and only if, (M, g) is isometric to the sphere (S^n, g_0) (cf. [4]).

On the other hand it is well–known, in the same hypothesis of C), that M is a homology sphere (that is all Betti numbers b_q, $0 < q < n$, vanish).

We observe that in all these results essential use is made of the assumption on the Ricci tensor.

The main purpose of this paper is to establish some results, of type C) for a compact $2p$–dimensional Riemannian manifold with positive scalar curvature

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without any assumption on the Ricci tensor. The main theorem that we obtain is the following.

THEOREM. *Let* (M, g) *be a compact Riemannian manifold of dimension* $2p$ *and scalar curvature* $\tau \geq 2p(2p-1)$. *Then*

$$ {}^{p}\lambda_1 \geq p(p+1) - (p^2-1)\,H/2 \,, \tag{1.1}$$

where

$$H = \sup \left\{ \frac{|C_{ijkh}\,\varphi^{ij}\,\varphi^{kh}|}{\varphi^{ij}\,\varphi_{ij}} : \varphi \in \Lambda^2(M) \right\}$$

is a measure of the deviation of M *from conformal flatness.*

Moreover for $p = 2, 3$ *if* (M, g) *is conformally flat (hence* $H = 0$*) and*

$$\chi(M)/\mathrm{vol}(M) \geq \chi(S^{2p})/\mathrm{vol}(S^{2p}) \,, \tag{1.2}$$

then equality holds in (1.1) *if, and only if,* (M, g) *is isometric to* (S^{2p}, g_0).

Remark 1.1. Theorem 1 entails that among all the compact conformally flat Riemannian manifolds M of dimension $2p$ (for $p = 2, 3$) whose scalar curvature is $\tau \geq 2p(2p-1)$ and $\chi(M)/\mathrm{vol}(M) \geq \chi(S^{2p})/\mathrm{vol}(S^{2p})$, the sphere (S^{2p}, g_0) is completely characterized by ${}^{p}\lambda_1$.

Moreover in section 4 we observe that the following holds.

PROPOSITION 1. *Let* (M, g) *be a compact Riemannian manifold of dimension* $2p$. *If the scalar curvature* $\tau > (p-1)(2p-1)\,H$, *then the p–th Betti number of* M *vanishes. If the scalar curvature* $\tau \geq (p-1)(2p-1)\,H$, *then for each p–form* $\varphi : \varphi$ *is harmonic if, and only if* φ *is parallel.*

Finally section 5 contributes a remark on $2p$–dimensional compact conformally flat Riemannian manifolds related to Yamabe's conjecture.

2. PRELIMINARIES

Let (M, g) be a Riemannian manifold of dimension n. We shall represent tensors by their components with respect to the natural base and shall use the summation convention. We denote by g_{ij}, $R^{i}{}_{jkh}$, $\rho_{ij} = R^{r}{}_{irj}$ the components of the metric, the curvature and the Ricci tensors respectively; ∇ stands for the operator of covariant differentiation (with respect to the Riemannian connection) and $\tau = g^{ij}\rho_{ij}$ denotes the scalar curvature. By $\chi(M)$ we denote the Euler-Poincaré characteristic of M.

The Weyl conformal curvature tensor $C = (C_{ijkh})$ is defined by

$$C_{ijkh} = R_{ijkh} - \frac{1}{n-2}(g_{ik}\,\rho_{jh} + g_{jh}\,\rho_{ik} - g_{ih}\,\rho_{jk} - g_{jk}\,\rho_{ih}) + \tag{2.1}$$

$$+ \frac{\tau}{(n-1)(n-2)}(g_{jh}\,g_{ik} - g_{jk}\,g_{ih})\,.$$

A Riemannian manifold (M, g) is called conformally flat if g is conformally related to a locally flat metric. It is known that for $n \geq 4$ M is conformally oat if $C = 0$. Moreover a conformally flat Riemannian manifold M is a space flf constant sectional curvature if, and only if, M is an Einstein space.

For p–forms

$$\varphi = \frac{1}{p!}\,\varphi_{i_1 i_2 \cdots i_p}\,dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_p}$$

and

$$\psi = \frac{1}{p!}\,\psi_{i_1 i_2 \cdots i_p}\,dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}\,,$$

the inner product (φ, ψ), the lengths $|\varphi|$ and $|\nabla\varphi|$ are defined by

$$(\varphi, \psi) = \frac{1}{p!}\,\varphi_{i_1 i_2 \cdots i_p}\,\psi^{i_1 i_2 \cdots i_p}\,,\quad |\varphi|^2 = (\varphi, \varphi)$$

$$|\nabla\varphi|^2 = \frac{1}{p!}\,\nabla_r\,\varphi_{i_1 i_2 \cdots i_p}\,\nabla^r\,\varphi^{i_1 i_2 \cdots i_p}\,.$$

By ${}^p\Delta$ we denote the Laplace operator acting on p–forms. If T is a tensor on M, we denote by $|T|$ the length of T.

3. PROOF OF THEOREM

For any q–forms φ we consider the so called Weizenböck's formula

$$\frac{1}{2}\,{}^0\Delta(|\varphi|^2) + |\nabla\varphi|^2 + Q_q(\varphi) = (\varphi, {}^q\Delta\,\varphi)\,, \tag{3.1}$$

where the quadratic form $Q_q(\varphi)$ is given by [4, p. 105]

$$Q_q(\varphi) = \frac{1}{(q-1)!}\left\{\frac{p-q}{p-1}\,\rho_{ij}\,\varphi^{ii_2 \cdots i_q}\,\varphi^{j}{}_{i_2 \cdots i_q} + \right. \tag{3.2}$$

$$\left. + \frac{(q-1)\,\tau}{2(p-1)(2p-1)}\,\varphi^{i_1 \cdots i_q}\,\varphi_{i_1 \cdots i_q} - \frac{q-1}{2}\,C_{ijkh}\,\varphi^{iji_3 \cdots i_q}\,\varphi^{kh}{}_{i_3 \cdots i_q}\right\}.$$

Moreover the following inequality holds for any q-form [3, p. 270]:

$$|\nabla \varphi|^2 \geq \frac{1}{q+1} |d\varphi|^2 + \frac{1}{2p-q+1} |\delta \varphi|^2 . \tag{3.3}$$

Now let φ be an eigenform for ${}^p\Delta$ corresponding to the eigenvalue ${}^p\lambda_1$, i.e. ${}^p\Delta\,\varphi = {}^p\lambda_1\,\varphi$ with $|\varphi| \neq 0$. Integration of (3.1) over M with respect to the volume element dv yields

$$ {}^p\lambda_1 \int_M |\varphi|^2 \, dv = \int_M |\nabla \varphi|^2 \, dv + \int_M Q_p(\varphi)\, dv . \tag{3.4}$$

Since $q = p$, $\tau \geq 2p(2p-1)$ and

$$- C_{ijkh}\, \varphi^{iji_3 \cdots i_p}\, \varphi^{kh}{}_{i_3 \cdots i_p} \geq - p!\, H\, |\varphi|^2 ,$$

we obtain from (3.2)

$$Q_p(\varphi) \geq p\,[p - (p-1)\, H/2]\, |\varphi|^2 . \tag{3.5}$$

Hence, (3.3), (3.4) and (3.5) yield

$$ {}^p\lambda_1 \int_M |\varphi|^2\, dv \geq \frac{1}{p+1} \int_M (|d\varphi|^2 + |\delta\varphi|^2)\, dv + p\,[p-(p-1)\,H/2] \int_M |\varphi|^2 dv =$$

$$= \frac{{}^p\lambda_1}{p+1} \int_M |\varphi|^2\, dv + p\,[p - (p-1)\, H/2] \int_M |\varphi|^2\, dv ,$$

which implies

$$ {}^p\lambda_1 \geq (p+1)\,[p - (p-1)\, H/2] .$$

Now we assume M to be conformally flat. Then $H = 0$ and hence

$$ {}^p\lambda_1 \geq p\,(p+1) .$$

Let φ a proper p-form corresponding to ${}^p\lambda_1 = p\,(p+1)$, then (3.4) becomes

$$\int_M (|\nabla \varphi|^2 - p\,|\varphi|^2)\, dv + \int_M (Q_p(\varphi) - p^2\, |\varphi|^2)\, dv = 0$$

where

$$|\nabla \varphi|^2 \geq p\, |\varphi|^2 \text{ and } Q_p(\varphi) \geq p^2\, |\varphi|^2 .$$

Thus $$\frac{p\,\tau\,|\,\varphi\,|^2}{2\,(2\,p-1)}=Q_p\,(\varphi)=p^2\,|\,\varphi\,|^2\,,\ \text{implying}$$

$$\tau=2p\,(2p-1)=\text{constant}\,.$$

We consider first the case $2p=4$. Then the Euler-Poincaré characteristic $\chi\,(M)$ of M is given by

$$\chi\,(M)=\frac{1}{32\,\Pi^2}\int_M (|\,R\,|^2-4\,|\,\varphi\,|^2+\tau^2)\,dv\,.$$

Since M is conformally flat, setting $C=0$ in (2.1), we get

$$\chi\,(M)=\frac{1}{32\,\Pi^2}\int_M\left(\frac{\tau^2}{6}-|\,\rho-\frac{\tau}{4}\,g\,|^2\right)dv\,.$$

Thus we have

$$\chi\,(M)\leq\frac{\tau^2\,\mathrm{vol}\,(M)}{192\,\Pi^2}=\frac{2\,\mathrm{vol}\,(M)}{\mathrm{vol}\,(S^4)}=\chi\,(S^4)\,\frac{\mathrm{vol}\,(M)}{\mathrm{vol}\,(S^4)}$$

where equality holds if and only if M is of constant sectional curvature 1. Then the condition (1.2) implies that M has constant sectional curvature 1. On the other hand it follows from Synge's theorem that M is simply connected. Therefore $(M\,,g)$ is isometric to $(S^4\,,g_0)$.

We assume now $2p=6$. In a 6-dimensional compact Riemannian manifold,

$$\chi\,(M)=\frac{1}{384\,\Pi^2}\int_M[\tau^3-12\,\tau\,|\,\rho\,|^2+3\,\tau\,|\,R\,|^2+16\,\rho^{ij}\,\rho_i^{\ k}\,\rho_{jk}+$$
$$+24\,\rho^{ij}\,\rho^{kh}\,R_{ijkh}-24\,\rho^{ij}\,R_i^{\ abc}\,R_{jabc}-8\,R^{ijkh}\,R_{i\ k}^{\ a\ b}\,R_{jahb}+$$
$$+4\,R^{ijkh}\,R_{ij}^{\ \ ab}\,R_{khab}]\,dv\,.$$

In the conformally flat case, setting $C=0$ in (2.1), we get

$$\chi\,(M)=\frac{1}{384\,\Pi^3}\int_M\left(\frac{21}{100}\,\tau^3-\frac{27}{20}\,\tau\,|\,\rho\,|^2+\frac{3}{2}\,\rho^{ij}\,\rho_i^{\ k}\,\rho_{jk}\right)dv= \tag{3.6}$$

$$=\frac{1}{384\,\Pi^3}\int_M\left(-\frac{3}{200}\,\tau^3-\frac{27}{20}\,\tau\,|\,\rho-\frac{\tau}{6}\,g\,|^2+\frac{3}{2}\,\rho^{ij}\,\rho_i^{\ k}\,\rho_{jk}\right)dv\,.$$

Moreover, the manifold M being conformally flat with constant scalar curvature $\tau = 30$, Lichnerowicz's formula

$$(3.7) \qquad \frac{1}{2} \Delta (| R |^2) = - | \nabla R |^2 - 4 R_{ikjh} \nabla_i \nabla_j \rho_{kh} - 2 \rho^{ij} R_i^{\ pqr} R_{jpqr} + \\ + 4 R^{ijkh} R_{i\ k}^{\ p\ q} R_{jphq} + R^{ijkh} R_{kh}^{\ \ pq} R_{ijpq} .$$

reduces to

$$(3.8) \qquad \frac{1}{2} \Delta (| R |^2) = - | \nabla \rho |^2 - \frac{3}{2} \rho^{ij} \rho_{jp} \rho^p{}_i + \frac{33}{2} | \rho |^2 - 1350$$

Integration of (3.8) yields

$$(3.9) \qquad \int_M | \nabla \rho |^2 dv + \frac{3}{2} \int_M \rho^{ij} \rho_{jp} \rho^p{}_i \, dv = \frac{33}{2} \int_M | \rho - 5 g |^2 dv + 1125 \operatorname{vol}(M) .$$

Hence, setting $\tau = 30$ in (3.6), (3.6) and (3.9) yield

$$(3.10) \qquad \chi (M) = - \frac{1}{384 \Pi^3} \int_M (| \nabla \rho |^2 + 24 | \rho - 5 g |^2) dv + \frac{15}{8\Pi^3} \operatorname{vol}(M) \leq \\ \leq \chi (S^6) \frac{15}{16\Pi^3} \operatorname{vol}(M) = \chi (S^6) \frac{\operatorname{vol}(M)}{\operatorname{vol}(S^6)}$$

where equality holds if, and only if, M has constant sectional curvature 1. At this point, the assumption of Theorem (3.10) and Synge's theorem imply that (M , g) is isometric to (S^6 , g_0).

Remark 3.1 In [5] S. Tachibana has shown that, in a $2p$–dimensional compact conformally flat Riemannian manifold with positive constant scalar curvature $\tau = 2p (2p - 1) k$, the proper space for p–forms corresponding to the minimal proper value $p (p + 1) k$ is the vector space of conformal Killing p–forms.

4. Proof of Proposition 1

If φ is a harmonic form of degree p, (3.1) entails the integral formula

$$(4.1) \qquad \int_M | \nabla \varphi |^2 dv + \int_M Q_p (\varphi) \, dv = 0 ,$$

and (3.2) yields

$$Q_p(\varphi) \geq \frac{p}{2}\left[\frac{\tau}{(2p-1)} - H(p-1)\right] |\varphi|^2 .$$

If $\tau > H(p-1)(2p-1)$, the quadratic form $Q_p(\varphi)$ is positive definite and by (4.1) we conclude that $Q_p(\varphi) = 0$, whereby $\varphi = 0$, and the p–th Betti number of M is zero.

If $\tau \geq H(p-1)(2p-1)$, then $Q_p(\varphi)$ is non-negative and by (4.1) we conclude that φ is parallel. The implication " φ parallel $\rightarrow \varphi$ harmonic " is well known.

Remark 4.1. Proposition 1 entails the well–known result whereby the p–th Betti number of a $2p$–dimensional compact conformally flat orientable Riemannian manifold with positive scalar curvature vanishes (cf. for example [5]).

5. A REMARK ON YAMABE'S CONJECTURE

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Following [1] we say that Yamabe's conjecture holds for (M, g) if there exists a metric g' conformal to g, which has scalar curvature $\tau' =$ constant.

In [1] Aubin has proved that Yamabe's conjecture is true in the following cases:

a) If (M, g) is not conformally flat and $n \geq 6$ (cf. [1] Theorem 9);

b) If (M, g) is conformally flat and the fundamental group of M is finite (cf. [1] Theorem 10).

In particular if the Ricci tensor is positive definite, then the fundamental group is finite (Myers).

Now we establish the following

PROPOSITION 2. *Let* (M, g) *be a compact conformally flat Riemannian manifold of dimension* $n = 2p \geq 4$.

b_1) *If the p–th Betti number of* M *is* $\neq 0$, *then Yamabe's conjecture is true.*

b_2) *If the fundamental group of* M *is finite, then* (M, g) *is conformal to the canonical sphere* (S^n, g_0).

Proof. b_1) By Yamabe's Theorem (cf. [1] p. 287) there exists a metric g' conformal to g whose scalar curvature τ' is either a constant ≤ 0 or a positive function. Since the p–th Betti number is $\neq 0$, Proposition 1 with $H = 0$, runs out the second possibility. Thus Yamabe's conjecture holds for (M, g).

b_2) By *b*) there exists a metric g' conformal to g with scalar curvature $\tau' =$ constant. In particular (M, g') is compact, conformally flat, with fundamental group finite and with constant scalar curvature. Then, by a Theorem of Tanno (cf. [7] Theorem A), (M, g') is of positive constant sectional curvature. It follows from Synge's theorem that (M, g') is isometric to a canonical sphere. Thus (M, g) is conformal to (S^n, g_0)

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The argument b_2) in the proof of Proposition 2 leads to the

COROLLARY. *If* (M, g) *is a compact Riemannian manifold of dimension* $2p \geq 4$ *whose fundamental group finite* $\neq \{0\}$, *then* g *is not conformally flat.*

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