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Characterization of some interpolation spaces (II)

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Analisi matematica. — *Characterization of some interpolation spaces* (II). Nota di ALESSANDRA LUNARDI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si caratterizzano alcuni spazi di interpolazione tra spazi di funzioni continue e domini di operatori ellittici del 2° ordine.

INTRODUCTION

In the study of evolution equations, it is sometimes useful to work in some interpolation space between the domain $D(A)$ of a differential operator A and the function space X in which $D(A)$ is embedded (see for instance Da Prato–Grisvard [1], Sinestrari [4], [5], Da Prato–Sinestrari [2]). Our kind of interpolation spaces are defined as follows: if $A : D(A) \subset X \to X$ is the infinitesimal generator of a semigroup e^{tA} in X, for every $\theta \in]0,1[$ set:

$$D_A(\theta) = \{x \in X \; ; \; \lim_{t \to 0^+} t^{-\theta}(e^{tA}x - x) = 0\} .$$

In this work we characterize $D_A(\theta)$ when X is a space of continuous functions and A is a strongly elliptic operator. More precisely, let Ω be an open set of $\mathbf{R}^n$ with regular boundary $\partial\Omega$. Then it is known (see Stewart [6]) that every strongly elliptic operator A with sufficiently regular coefficients is the infinitesimal generator of an analytic semigroup in $C_0^0(\overline{\Omega})$, the space of all continuous functions in $\overline{\Omega}$ which vanish on $\partial\Omega$. We prove that, if $\theta \neq \frac{1}{2}$, then $D_A(\theta) = h^{2\theta}(\overline{\Omega}) \cap C_0^0(\overline{\Omega})$, where $h^{2\theta}(\overline{\Omega})$ is the space of little Hölder continuous functions of order 2θ. If $2\theta < 1$, then $h^{2\theta}(\overline{\Omega})$ is defined as the subspace of $C^{2\theta}(\overline{\Omega})$ consisting of all f such that

$$\lim_{\tau \to 0^+} \sup_{\substack{x,y \in \overline{\Omega} \\ |x-y| \leq \tau}} \frac{|f(x) - f(y)|}{\tau^{2\theta}} = 0 ,$$

whereas, if $2\theta > 1$, $h^{2\theta}(\overline{\Omega})$ is defined as the space of all $f \in C^2(\overline{\Omega}) \cap C_0^0(\overline{\Omega})$ such that $\frac{\partial f}{\partial x_i} \in h^{2\theta-1}(\overline{\Omega})$ $\forall i = 1, \cdots, n$. A similar result holds if we take $X = C^0(\overline{\Omega})$ (without requiring boundary conditions).

(*) Nella seduta del 25 giugno 1982.

1. Definitions and some properties of interpolation spaces

Throughout this section X and Y will denote two Banach spaces, with Y continuously embedded in X (we shall write $Y \hookrightarrow X$).

Definition 1.1. For every $\theta \in]0, 1[$ set:

$$C(\theta ; Y, X) = \{u :]0, 1] \to X ; t \to t^{\theta} u(t) \in C([0, 1] ; Y), t \to t^{\theta} u'(t) \in \\ \in C([0, 1] ; X), \lim_{t \to 0^+} \|t^{\theta} u(t)\|_Y = \lim_{t \to 0^+} \|t^{\theta} u'(t)\|_X = 0\}.$$

$C(\theta ; Y, X)$ is a Banach space under the norm:

$$\|u\|_{\theta; Y, X} = \|t^{\theta} u(t)\|_{L^{\infty}(0,1; Y)} + \|t^{\theta} u'(t)\|_{L^{\infty}(0,1; X)}.$$

It is easy to show that if $u \in C(\theta ; Y, X)$ then there exists

$$X - \lim_{t \to 0^+} u(t) = u(0).$$

Definition 1.2. For every $\theta \in]0, 1[$ set:

$$(Y, X)_{\theta} = \{u(0), u \in C(\theta ; Y, X)\}.$$

$(Y, X)_{\theta}$ is a Banach space under the norm:

$$\|x\|_{\theta} = \inf_{\substack{u(0)=x \\ u \in C(\theta; Y, X)}} \|u\|_{\theta; Y, X}$$

and it can be shown that Y is dense in $(Y, X)_{\theta}$ (see Sinestrari [3]).

In Section 2 we shall also use the following characterization of $(Y, X)_{\theta}$:

Proposition 1.3. *$(Y, X)_{\theta}$ is the set of all $x \in X$ such that*:

$$x = u(t) + v(t) \qquad \forall t \in]0, 1]$$

where:

$$\text{(1–2)} \qquad \begin{cases} t \to t^{\theta} u(t) \in C([0, 1] ; Y) \\ t \to t^{\theta - 1} v(t) \in C([0, 1] ; X) \end{cases}$$

and the norm:

$$\|x\| = \inf_{\substack{u+v=x \\ u, v \text{ satisfy (1–2)}}} (\|t^{\theta} u\|_{L^{\infty}(0,1; Y)} + \|t^{\theta - 1} v\|_{L^{\infty}(0,1; X)})$$

is equivalent to the norm of $(Y, X)_{\theta}$.

In the case $Y = D(A)$, where A is the infinitesimal generator of a semigroup e^{tA} in X, there are other useful characterizations of $(Y, X)_{\theta}$. Define, for every $\theta \in]0,1[$:

$$D_A(\theta) = \{x \in X \; ; \; \lim_{t\to 0^+} t^{-\theta}(e^{tA}x - x) = 0\},$$

$$\|x\|_{D_A(\theta)} = \|x\|_X + \|t^{-\theta}(e^{tA}x - x)\|_{L^\infty(0,1;X)}.$$

Then $D_A(\theta)$ is a Banach space under the norm $\|\cdot\|_{D_A(\theta)}$ and the following characterization holds:

PROPOSITION 1.4. *Under the above assumptions, for every* $\theta \in]0,1[$ *we have*:

$$D_A(\theta) \cong (D(A), X)_{1-\theta}$$

if $D(A)$ *is endowed with the graph norm.*

The proofs of Propositions 1.3 and 1.4 can be found in Da Prato–Grisvard [1].

2. CHARACTERIZATION OF SOME INTERPOLATION SPACES

Let Ω be open in $\mathbf{R}^n$ and set:

$$C_0^0(\overline{\Omega}) = \{f \in C^0(\overline{\Omega}) \; ; \; f(x) = 0 \;\; \forall x \in \partial\Omega \, , \lim_{|x|\to+\infty} f(x) = 0 \text{ if } \Omega \text{ is unbounded}\},$$

$$C_0^2(\overline{\Omega}) = \{f \in C^2(\overline{\Omega}) \; ; \; \forall \, |\alpha| = 0, 1, 2, \;\; D^\alpha f \in C_0^0(\overline{\Omega})\}.$$

These spaces are endowed with their natural norms.

The principal result of this paper is the characterization of the interpolation space $D_A(\theta)$ between $C^0(\overline{\Omega})$ (resp. $C_0^0(\overline{\Omega})$) and the domain of an elliptic operator A in $C^0(\overline{\Omega})$ (resp. $C_0^0(\overline{\Omega})$). To this purpose we must first consider the cases $\Omega = \mathbf{R}^n$ and $\Omega = \mathbf{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbf{R}^n \; ; \; x_n > 0\}$ and characterize the interpolation spaces between $C^0(\overline{\Omega})$ and $C^2(\overline{\Omega})$ (resp. $C_0^0(\overline{\Omega})$ and $C_0^2(\overline{\Omega})$). We start by giving the following definition:

DEFINITION 2.1. For every $\sigma \in]0,1[$ set:

$$h^\sigma(\overline{\Omega}) = \{f \in C^0(\overline{\Omega}) \; ; \; \lim_{\tau\to 0^+} \sup_{\substack{x,y\in\overline{\Omega} \\ |x-y|\le\tau}} \tau^{-\sigma} |f(x) - f(y)| = 0\},$$

$$h^{\sigma+1}(\overline{\Omega}) = \left\{f \in C^1(\overline{\Omega}) \; ; \; \frac{\partial f}{\partial x_i} \in h^\sigma(\overline{\Omega}) \;\; \forall i = 1, \cdots, n\right\},$$

$$h_0^\sigma(\overline{\Omega}) = h^\sigma(\overline{\Omega}) \cap C_0^0(\overline{\Omega}),$$

$$h_0^{\sigma+1}(\overline{\Omega}) = \left\{f \in C^1(\overline{\Omega}) \cap C_0^0(\overline{\Omega}) \; ; \; \frac{\partial f}{\partial x_i} \in h_0^\sigma(\overline{\Omega}) \;\; \forall i = 1, \cdots, n\right\}.$$

Then $h^\sigma(\overline{\Omega})$ and $h_0^\sigma(\overline{\Omega})$ (resp $h^{\sigma+1}(\overline{\Omega})$ and $h_0^{\sigma+1}(\overline{\Omega})$) are Banach spaces under the C^σ-norm (resp. under the $C^{\sigma+1}$-norm).

PROPOSITION 2.2. *For every* $\theta \in]0, 1[$, $\theta \neq \frac{1}{2}$, *we have*:

$$(C_0^2(\mathbf{R}^n), C_0^0(\mathbf{R}^n))_{1-\theta} \cong h_0^{2\theta}(\mathbf{R}^n),$$

$$(C_0^2(\mathbf{R}_+^n), C_0^0(\mathbf{R}_+^n))_{1-\theta} \cong h_0^{2\theta}(\mathbf{R}_+^n).$$

Sketch of the proof: First we characterize $C_0^2(\mathbf{R}^n)$ (resp. $C_0^2(\mathbf{R}_+^n)$) as the intersection of the domains of simple differential operators (the second derivatives) and then we use Proposition 1, p. 88 of Triebel [7], which can be easily adapted to our situation.

PROPOSITION 2.3. *Let* Ω *be a bounded open set of* $\mathbf{R}^n$ *with* $\partial\Omega$ *of class* C^2. *Then for every* $\theta \in]0, 1[$, $\theta \neq \frac{1}{2}$, *we have*:

$$(C^2(\overline{\Omega}) \cap C_0^0(\overline{\Omega}), C^0(\overline{\Omega}))_{1-\theta} \cong h^{2\theta}(\overline{\Omega}) \cap C_0^0(\overline{\Omega}),$$

$$(C_0^2(\overline{\Omega}), C_0^0(\overline{\Omega}))_{1-\theta} \cong h_0^2(\overline{\Omega}).$$

The proof is based on a method of localization which uses Proposition 2.2.

Under the hypotheses of Proposition 2.3, suppose that $\partial\Omega$ is of classe $C^{2+\mu}$ for some $\mu > 0$ and set:

$$\begin{cases} X = C^0(\overline{\Omega}) \\ D(A) = \{f \in C_0^0(\overline{\Omega})\ ;\ Af \in C^0(\overline{\Omega})\} \\ Af = \sum_{i,j=1}^{n} a_{ij} \dfrac{\partial^2 f}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} b_i \dfrac{\partial f}{\partial x_i} + cf \qquad \forall f \in D(A), \end{cases}$$

$$\begin{cases} X_0 = C_0^0(\overline{\Omega}) \\ D(A_0) = \{f \in C_0^0(\overline{\Omega})\ ;\ Af \in C_0^0(\overline{\Omega})\} \\ A_0 f = Af \qquad \forall f \in D(A_0), \end{cases}$$

where $a_{ij}, b_i, c \in C^0(\overline{\Omega})$ and there exists $\nu > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i\, \xi_j \geq \nu |\xi|^2$ $\forall x \in \overline{\Omega}$, $\forall \xi \in \mathbf{R}^n$.

Under all these assumptions, $A : D(A) \to X$ is the infinitesimal generator of an analytic semigroup in X and $A_0 : D(A_0) \to X_0$ is the infinitesimal generator of a strongly continuous analytic semigroup in X_0 (see Stewart [6]).

The main result of this paper is the following:

THEOREM 2.4. *Under the above assumptions, for every* $\theta \in]0, 1[$, $\theta \neq \frac{1}{2}$, *we have*:

$$D_A(\theta) \cong (D(A), C^0(\overline{\Omega}))_{1-\theta} \cong h^{2\theta}(\overline{\Omega}) \cap C_0^0(\overline{\Omega}), \tag{2.4}$$

$$D_{A_0}(\theta) \cong (D(A_0), C_0^0(\overline{\Omega}))_{1-\theta} \cong h^{2\theta}(\overline{\Omega}) \cap C_0^0(\overline{\Omega}). \tag{2.5}$$

Here we sketch the proof of (2.4). To prove $\hookleftarrow$, it is sufficient to observe that $C^2(\overline{\Omega}) \cap C^0_0(\overline{\Omega}) \hookrightarrow D(A)$, and hence we have:

$$C^0_0(\overline{\Omega}) \cap h^{2\theta}(\overline{\Omega}) \cong (C^2(\overline{\Omega}) \cap C^0_0(\overline{\Omega}), C^0(\overline{\Omega}))_{1-\theta} \hookrightarrow$$

$$\hookrightarrow (D(A), C^0(\overline{\Omega}))_{1-\theta} \cong D_A(\theta).$$

Now we prove the inclusion $\hookrightarrow$. Let $f \in D_A(\theta)$, then $f = u(t) + v(t)\ \forall t \in]0,1]$ where u and v satify (1.2). For every $t \in]0,1]$ there exists an extension $U(t)$ of $u(t)$ to $\mathbf{R}^n$ and an extension $V(t)$ ov $v(t)$ to $\mathbf{R}^n$ such that U and V satisfy (1.2) with $X = C^0_0(\mathbf{R}^n)$, $Y = C^1_0(\mathbf{R}^n) = \left\{ f \in C^1(\mathbf{R}^n)\,;\ \lim\limits_{|x| \to +\infty} f(x) = \lim\limits_{|x| \to +\infty} \frac{\partial f}{\partial x_i}(x) = 0 \right.$ $\left. \forall i = 1, \cdots, n \right\}$ and moreover $U(t) + V(t)$ is constant. Extend f to $\mathbf{R}^n$, setting:

$$F(x) = U(t)(x) + V(t)(x) \qquad \forall x \in \mathbf{R}^n,\ \forall t \in]0,1].$$

For every $t \in]0,1]$ let Q_t be the cube centered at $0 \in \mathbf{R}^n$ with edge t, and set:

$$M_t(\varphi)(x) = \int_{x+Q_t} \varphi(y)\,dy \qquad \forall \varphi \in C^1(\mathbf{R}^n).$$

Then, setting:

$$W(t)(x) = M_t(U(t))(x) \qquad \forall x \in \mathbf{R}^n,\ \forall t \in]0,1]$$

we can write F as:

$$F = W(t) + V(t) + (U(t) - W(t)) = W(t) + G(t) \qquad \forall t \in]0,1]$$

where W and G satisfy:

$$t \to t^{1-\theta}\, W(t) \in C([0,1]\,; C^2_0(\mathbf{R}^n)),$$

$$t \to t^{-\theta}\, G(t) \in C([0,1]\,; C^0_0(\mathbf{R}^n)).$$

Then F belongs to $(C^2_0(\mathbf{R}^n), C^0_0(\mathbf{R}^n))_{1-\theta} = h^{2\theta}_0(\mathbf{R}^n)$, and hence $F|_{\overline{\Omega}} = f \in h^{2\theta}(\overline{\Omega})$. Moreover, as $D(A)$ is dense in $D_A(\theta)$, we have: $f(x) = 0\ \forall x \in \partial\Omega$, so $\hookrightarrow$ of (2.4) is proved. The proof of (2.5) is analogous.

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