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Analisi matematica. — *Existence and regularity results for abstract non autonomous parabolic equations.* Nota di PAOLO ACQUISTAPACE (*) e BRUNELLO TERRENI (**), presentata (***) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studiano esistenza, unicità e regolarità delle soluzioni strette, classiche e forti dell'equazione di evoluzione non autonoma $u'(t) - A(t)u(t) = f(t)$, con il dato iniziale $u(0) = x$, in spazi di Banach. I dominii degli operatori $A(t)$ variano in t e non sono necessariamente densi in E. Si danno condizioni necessarie e sufficienti per l'esistenza e la regolarità hölderiana della soluzione e della sua derivata.

0. INTRODUCTION

Let E be a Banach space, $\{A(t)\}_{t\in[0,T]}$ a family of closed linear operators on E. Suppose that for each $t\in[0,T]$, $A(t)$ generates an analytic semigroup and has a domain $D(A(t))$ varying with t and not necessarily dense in E. We consider the following Cauchy problem:

$$(\mathrm{P})\qquad \begin{cases} u'(t) - A(t)u(t) = f(t), & t\in[0,T] \\ u(0) = x & \\ x\in E \quad , \quad f\in C([0,T],E) & \text{prescribed.} \end{cases}$$

Many authors have studied this problem (see the references). In this paper we assume the classical hypotheses of Kato-Tanabe [8]. Precisely, we suppose:

I) For each $t\in[0,T]$, $A(t)$ is a closed linear operator on the Banach space E with domain $D(A(t))$, generating an analytic semigroup $\{e^{\xi A(t)}\}_{\xi\geq 0}$; in particular:

(*i*) there exists $\theta_0\in\,]\pi/2,\pi]$ such that

$$\rho(A(t)) \supseteq \Sigma_{\theta_0} \triangleq \{z\in\mathbf{C} : z = \rho e^{i\theta}, \rho\in[0,+\infty[\,,\ \theta\in]-\theta_0,\theta_0[\} \quad \forall t\in[0,T];$$

(*ii*) there exists $M > 0$ such that the resolvent operator $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ satisfies

$$\|R(\lambda, A(t))\|_{\mathscr{L}(E)} \leq M/|\lambda| \qquad \forall\lambda\in\Sigma_{\theta_0} \quad , \quad \forall t\in[0,T].$$

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II) The operator-valued function $t \to R(\lambda, A(t))$ is continuously differentiable in $\mathscr{L}(E)$ for each $\lambda \in \Sigma_{\theta_0}$; moreover there exist $L > 0$ and $\alpha \in]0, 1]$ such that

$$\left\| \frac{\partial}{\partial t} R(\lambda, A(t)) \right\|_{\mathscr{L}(E)} \leq L/|\lambda|^{\alpha} \qquad \forall \lambda \in \Sigma_{\theta_0} \quad , \quad \forall t \in [0, T].$$

III) There exist $B > 0$ and $\eta \in]0, 1]$ such that

$$\left\| \frac{d}{dt} A(t)^{-1} - \frac{d}{d\tau} A(\tau)^{-1} \right\|_{\mathscr{L}(E)} \leq B\, |t - \tau|^{\eta} \qquad \forall t, \tau \in [0, T].$$

Let us specify now what we mean by a solution of problem (P). Denote by $C([0, T], E)$ (resp. $C(]0, T], E)$) the class of continuous functions $[0, T] \to E$ (resp. $]0, T] \to E$), and set

$$\begin{aligned} D = \{u \in C([0, T], E) : u(t) \in D(A(t))\ \forall t \in [0, T] \\ \text{and} \quad t \to A(t)\, u(t) \qquad \text{is in} \quad C([0, T], E)\}, \\ D_0 = \{u \in C([0, T], E) : u(t) \in D(A(t))\ \forall t \in]0, T] \\ \text{and} \quad t \to A(t)\, u(t) \qquad \text{is in} \quad C(]0, T], E)\}. \end{aligned}$$

Now we define our solutions.

DEFINITION 0.1. $u : [0, T] \to E$ is a *strict solution* of Problem (P) if $u \in D$ and

$$u'(t) - A(t)\, u(t) = f(t) \qquad \forall t \in [0, T] \quad , \quad u(0) = x.$$

DEFINITION 0.2. $u : [0, T] \to E$ is a *classical solution* of Problem (P) if $u \in D_0$ and

$$u'(t) - A(t)\, u(t) = f(t) \qquad \forall t \in]0, T] \quad , \quad u(0) = x.$$

DEFINITION 0.3. $u : [0, T] \to E$ is a *strong solution* of Problem (P) if $u \in C([0, T], E)$ and there exists $\{u_n\}_{n \in N} \subseteq D$ such that:

(*i*) $u_n \to u$ in $C([0, T], E)$;

(*ii*) $u_n'(t) - A(t)\, u_n(t) \triangleq f_n(t) \in C([0, T], E)$ and $f_n \to f$ in $C([0, T], E)$;

(*iii*) $u_n(0) \to x$ in E.

Remark 0.4. Assumption III could be weakened (see Yagi [15]), without affecting our results, but we prefer to assume it since it allows much simpler proofs and in concrete situations it seems easier to be handled than Yagi's condition.

Remark 0.5. It is clear from the definitions that every strict solution is also a classical one and a strong one, whereas it is not true, in general, that every classical solution is also a strong solution. This is true, however, under hypotheses I and II, provided $D(A(0))$ is dense in E (see Remark 2.6). On the other hand it is easy to verify that every strict, or classical, or strong solution of Problem (P) is also a weak solution in the sense of Kato-Tanabe [8].

In this paper we give existence, uniqueness and regularity results for strict, classical and strong solutions of Problem (P). In addition we give a representation formula for the solution, without passing through the construction of the fundamental solution; our method is mainly inspired by the techniques of Da Prato-Grisvard [4]. Our formula can be heuristically derived from the following argument: we look for a solution of Problem (P) of this kind

$$u(t) \triangleq e^{tA(t)}\, x + \int_0^t e^{(t-s)A(t)}\, g(s)\, ds, \qquad t \in [0, T]. \tag{0.1}$$

Of course when $A(t) = A =$ constant, this formula with $g = f$ gives the ordinary mild solution of Problem (P); thus in this case g may be considered as a " perturbation " of f. Taking the formal derivative of (0.1) we get

$$u'(t) = A(t)\, u(t) + \left[g(t) + \int_0^t \left[\frac{\partial}{\partial t}\, e^{\xi A(t)} \right]_{\xi = t-s} g(s)\, ds + \left[\frac{\partial}{\partial t}\, e^{\xi A(t)} \right]_{\xi = t} x \right].$$

Hence if we want (0.1) to be a solution of Problem (P), we must choose g such that

$$g(t) + \int_0^t P(t, s)\, g(s)\, ds + P(t, 0)\, x = f(t), \tag{0.2}$$

$$P(t, s) \triangleq \left[\frac{\partial}{\partial t}\, e^{\xi A(t)} \right]_{\xi = t-s}.$$

Denote by P the integral operator

$$P\varphi(t) \triangleq \int_0^t P(t, s)\, \varphi(s)\, ds. \tag{0.3}$$

Then the representation formula for the solution of problem (P) is formally given by

$$u(t) = e^{tA(t)}\, x + \int_0^t e^{(t-s)A(t)} \left[(1 + P)^{-1} (f - P(\cdot, 0)\, x) \right](s)\, ds, \qquad t \in [0, T]. \tag{F}$$

We also study the maximal regularity of the strict solution: we say that there is maximal regularity for the solution of Problem (P) if it has Hölder continuous derivative for some exponent $\alpha \in]0,1]$ whenever f is Hölder continuous with the same α, provided the vectors x and $f(0)$ satisfy some suitable compatibility conditions. Here we get a necessary and sufficient condition on x and $f(0)$ which generalizes the results of [9], [11], [12], [6]. More details and proofs of our results can be found in a forthcoming paper.

1. PRELIMINARIES

DEFINITION 1.1. Let $\beta \in]0,1[$, A be a closed linear operator on the Banach space E, satisfying hypothesis I. Let $x \in E$; x is said to be in $D_A(\beta,\infty)$ if there exists $u \in C^1(]0,+\infty[,E) \cap C(]0,+\infty[,D(A))$ such that:

(*i*) $\|t^{1-\beta}u(t)\|_E + \|t^{1-\beta}u'(t)\|_E + \|t^{1-\beta}Au(t)\|_E \leq \text{constant} \quad \forall t \in]0,+\infty[$,

(*ii*) $u(0) = x$.

Condition (*ii*) is meaningful since (*i*) implies $u \in C^{1-\beta}([0,+\infty[,E)$.

PROPOSITION 1.2. *Under the hypotheses of Definition 1.1 we have:*

(*i*) $D(A) \subseteq D_A(\beta,\infty) \subseteq \overline{D(A)} \qquad \forall \beta \in]0,1[$;

(*ii*)
$$D_A(\beta,\infty) = \{x \in E : \sup_{t>0} t^{1-\beta}\|e^{tA}x - x\|_E < \infty\} = $$
$$= \{x \in E : \sup_{\lambda>0} \lambda^\beta \|AR(\lambda,A)x\|_E < \infty\}.$$

Now we go back to our situation and, to start with, we represent the analytic semigroup $\{e^{\xi A(t)}\}_{\xi\geq 0}$ by a Dunford integral.

Let γ be an arbitrary continuous simple path contained in Σ_{θ_0}, joining $+\infty e^{-i\theta}$ and $+\infty e^{i\theta}$, $\theta \in]\pi/2,\pi]$ being fixed. For convenience, we choose from now on

$$\gamma = \gamma_0 \cup \gamma_+ \cup \gamma_-,$$

where

$$\gamma_0 \triangleq \{\lambda \in \mathbf{C} : |\lambda| = 1, |\arg\lambda| \leq \theta\},$$
$$\gamma_\pm \triangleq \{\lambda \in \mathbf{C} : \lambda = \rho e^{\pm i\theta}, \rho \geq 1\}.$$

Then under hypothesis I we have

$$e^{\xi A(t)} = \frac{1}{2\pi i}\int_\gamma e^{\xi\lambda} R(\lambda, A(t))\, d\lambda, \qquad \forall \xi > 0 \quad , \quad \forall t \in [0,T],$$

and under hypotheses I and II

$$(1.1) \qquad \mathrm{P}(t,s) = \left[\frac{\partial}{\partial t} e^{\xi \mathrm{A}(t)}\right]_{\xi=t-s} = \frac{1}{2\pi i}\int_{\gamma} e^{\lambda(t-s)} \frac{\partial}{\partial t} \mathrm{R}(\lambda, \mathrm{A}(t))\, \mathrm{d}\lambda\,, \qquad 0 \le s < t \le \mathrm{T}\,,$$

the integrals being absolutely convergent.

In addition we have

$$(1.2) \qquad \|\mathrm{P}(t,s)\|_{\mathscr{L}(\mathrm{E})} \le \mathrm{K}/(t-s)^{1-\alpha}, \qquad 0 \le s < t \le \mathrm{T}\,.$$

Hence it is easy to show that the operator P defined in (0.3) is continuous on $\mathrm{L}^p(0, \mathrm{T}; \mathrm{E})$, $p \in [1, +\infty]$; moreover $(1+\mathrm{P})^{-1}$ exists and is also continuous on such spaces. As a direct consequence we have:

PROPOSITION 1.3. *Under hypotheses I and II, let* $x \in \mathrm{E}$ *and* $f \in \mathrm{C}([0, \mathrm{T}], \mathrm{E})$; *then formula* (F) *defines a vector-valued function* $u \in \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{E}) \cap \mathrm{C}(]0, \mathrm{T}], \mathrm{E})$. *Moreover if* $x \in \overline{\mathrm{D}(\mathrm{A}(0))}$ *then* $u \in \mathrm{C}([0, \mathrm{T}], \mathrm{E})$.

Formula (F) plays a key role in proving existence and regularity results for classical, strict and strong solutions of Problem (P).

Another representation formula, similar to (F), is used to obtain our uniqueness results. Define

$$\tilde{\mathrm{P}}\varphi(t) \triangleq \int_0^t \tilde{\mathrm{P}}(t,s)\,\varphi(s)\,\mathrm{d}s \quad , \quad \tilde{\mathrm{P}}(t,s) \triangleq \left[\frac{\partial}{\partial s} e^{\xi \mathrm{A}(s)}\right]_{\xi=t-s}, \qquad 0 \le s < t \le \mathrm{T};$$

then $\tilde{\mathrm{P}}$ has the same properties of P, and we have:

PROPOSITION 1.4. *Under hypotheses I and II, suppose u is a strict solution of Problem* (P). *Then*

$$(\tilde{\mathrm{F}}) \qquad u(t) = (1-\tilde{\mathrm{P}})^{-1}\left[e^{t\mathrm{A}(0)}\,x + \int_0^t e^{(t-s)\mathrm{A}(s)} f(s)\,\mathrm{d}s\right], \qquad t \in [0, \mathrm{T}]\,.$$

COROLLARY 1.9 (a priori estimate, uniqueness). *Under hypotheses I and II let u be a strict* (*or classical, or strong*) *solution of Problem* (P). *Then*

$$\|u(t)\|_{\mathrm{E}} \le \text{constant} \cdot \left[\|x\|_{\mathrm{E}} + \int_0^t \|f(s)\|_{\mathrm{E}}\,\mathrm{d}s\right], \qquad \forall t \in [0, \mathrm{T}]\,.$$

Hence the strict (or classical, or strong) solution of Problem (P) is unique.

2. MAIN RESULTS

Here is a list of our main results, whose proofs can be found in a forthcoming paper.

THEOREM 2.1 (necessary conditions). *Under hypotheses I, II let* $x \in \mathrm{E}$ *and* $f \in \mathrm{C}([0, \mathrm{T}], \mathrm{E})$.

(*i*) *If u is a classical solution of* (P), *then* $x \in \overline{\mathrm{D}(\mathrm{A}(0))}$.

(*ii*) *If u is a strict solution of* (P), *then* $x \in \mathrm{D}(\mathrm{A}(0))$ *and* x, $f(0)$ *satisfy*

$$\mathrm{A}(0)\,x + f(0) - \left[\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{A}(t)^{-1}\right]_{t=0} \mathrm{A}(0)\,x \in \overline{\mathrm{D}(\mathrm{A}(0))}\,. \tag{2.1}$$

(*iii*) *If u is a strong solution of* (P), *then* $x \in \overline{\mathrm{D}(\mathrm{A}(0))}$ *and* x, $f(0)$ *satisfy*

$$\begin{cases} \exists\, \{x_n\}_{n \in \mathbf{N}} \subseteq \mathrm{D}(\mathrm{A}(0)) \quad , \quad \{y_n\}_{n \in \mathbf{N}} \subseteq \mathrm{E} \quad \textit{such that} \\ x_n \to x \quad \text{in E} \quad , \quad y_n \to f(0) \quad \text{in E}, \quad \textit{and} \\ \mathrm{A}(0)\,x_n + y_n - \left[\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{A}(t)^{-1}\right]_{t=0} \mathrm{A}(0)\,x_n \in \overline{\mathrm{D}(\mathrm{A}(0))}\,. \end{cases} \tag{2.2}$$

THEOREM 2.2 (classical solutions). *Under hypotheses I, II, III let* $x \in \overline{\mathrm{D}(\mathrm{A}(0))}$ *and* $f \in \mathrm{C}([0, \mathrm{T}], \mathrm{E}) \cap \mathrm{C}^{\sigma}(]0, \mathrm{T}], \mathrm{E})$, $\sigma \in]0, 1]$; *then the function* $u(t)$ *defined by* (F) *is the classical solution of* (P). *Moreover* $u \in \mathrm{C}^{1,\sigma \wedge \delta}(]0, \mathrm{T}], \mathrm{E})$ *for each* $\delta \in]0, \eta[\cap]0, \alpha]$.

THEOREM 2.3 (strict solutions). *Under hypotheses I, II, III let* $x \in \mathrm{D}(\mathrm{A}(0))$, $f \in \mathrm{C}^{\sigma}([0, \mathrm{T}], \mathrm{E})$ *with* $\sigma \in]0, 1]$, *and suppose* x, $f(0)$ *satisfy condition* (*2.1*). *Then the function* $u(t)$ *defined by* (F) *is the strict solution of* (P). *Moreover* $u \in \mathrm{C}^{1,\delta \wedge \sigma}(]0, \mathrm{T}], \mathrm{E})$ *for each* $\delta \in]0, \eta[\cap]0, \alpha]$.

THEOREM 2.4 (maximal regularity). *Under hypotheses I, II, III let* $x \in \mathrm{D}(\mathrm{A}(0))$, $f \in \mathrm{C}^{\delta}([0, \mathrm{T}], \mathrm{E})$ *with* $\delta \in]0, \eta[\cap]0, \alpha]$, *and suppose u is a strict solution of Problem* (P). *Then* $u \in \mathrm{C}^{1,\delta}([0, \mathrm{T}], \mathrm{E})$ *if, and only if, the vectors x and* $f(0)$ *satisfy*

$$\mathrm{A}(0)\,x + f(0) - \left[\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{A}(t)^{-1}\right]_{t=0} \mathrm{A}(0)\,x \in \mathrm{D}_{\mathrm{A}(0)}(\delta, \infty)\,. \tag{2.3}$$

THEOREM 2.5 (strong solutions). *Under hypotheses I, II let* $x \in \overline{\mathrm{D}(\mathrm{A}(0))}$ *and* $f \in \mathrm{C}([0, \mathrm{T}], \mathrm{E})$, *and suppose that* x, $f(0)$ *satisfy condition* (*2.2*). *Then the function* $u(t)$ *defined by* (F) *is the strong solution of Problem* (P).

Moreover $u \in C^{\delta}(]0, T], E)$ for each $\delta \in]0, 1]$ and:

(*i*) if $\beta \in]0, \alpha[$, then $u \in C^{\beta}([0, T], E)$ if and only if $x \in D_{A(0)}(\beta, \infty)$;

(*ii*) if $\beta \in [\alpha, 1[$ and $x \in D_{A(0)}(\beta, \infty)$, then $u \in C^{\delta}([0, T], E)$ for each $\delta \in]0, \beta[$.

Remark 2.6. Existence of strict and classical solutions is still guaranteed if, under the same assumptions on x, we suppose that $f \in C([0, T], E)$ and there exists $t^0 \in]0, T]$ such that

$$\int_0^{t^0} [\omega(t)/t] \, dt < \infty ,$$

where $\omega(t)$ is the modulus of continuity of f. This generalizes a result of Crandall-Pazy [3]. On the other hand the only assumption $f \in C([0, T], E)$ is not sufficient, even if $x = 0$ and $A(t) = A$ (see [1], [14], [5]).

Remark 2.7. A classical solution needs not be also a strong solution, since all hypotheses of Theorem 2.2 can hold without condition (2.2) being true: however if for instance $D(A(0))$ is dense in E, then (2.2) certainly holds for any $x \in E$ and $f \in C([0, T], E)$, and in this case under hypotheses I and II every classical solution is a strong one.

Here is another condition, stronger than (2.2), which seems easier to handle in concrete examples. Namely, condition (2.2) holds for any $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$ provided

$$\text{range of } \left[\frac{d}{dt} A(t)^{-1}\right]_{t=0} \subseteq \overline{D(A(0))}; \tag{2.4}$$

this is the case if, for instance, $D(A(t))$ does not depend on t, or more generally if $D(A(t)) \subseteq \overline{D(A(0))}$ for each $t \in [0, T]$.

Finally we note that the preceding results apply to many concrete situations; two examples are described in our next paper.

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