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A Note on Regular-closed Functions

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Topologia. — *A Note on Regular-closed Functions.* Nota di TAKASHI NOIRI presentata (*) dal Socio E. MARTINELLI.

RIASSUNTO. — Se X ed Y sono spazi topologici, una funzione $f : X \to Y$ è detta regolarmente chiusa [5] se essa trasforma ogni insieme regolarmente chiuso di X in un insieme chiuso di Y. Si dimostra che una funzione regolarmente chiusa $f : X \to Y$ risulta chiusa se X è normale.

1. INTRODUCTION

In [5], P. E. Long and L. L. Herrington have defined a function $f : X \to Y$ to be *regular-closed* if for each regular closed set A of X, $f(A)$ is closed in Y and they have obtained numerous properties of such functions. In [8], regular-closed functions are called *almost-closed.* The purpose of the present note is to improve some of the theorems given in [5]. Throughout the present Note, spaces will always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : X \to Y$ will denote a function (not necessarily continuous) f of a space X into a space Y. Let S be a subset of a space X. The closure of S and the interior of S in X are denoted by $\mathrm{Cl}_X(S)$ and $\mathrm{Int}_X(S)$, respectively. A subset S of X is said to be *regular closed* (*regular open*) if $\mathrm{Cl}_X(\mathrm{Int}_X(S)) = S$ (resp. $\mathrm{Int}_X(\mathrm{Cl}_X(S)) = S$).

2. REGULAR-CLOSED FUNCTIONS

A subset S of a space X is said to be *semi-open* [4] if there exists an open set U of X such that $U \subset S \subset \mathrm{Cl}_X(U)$. The family of all semi-open sets in X is denoted by SO (X). A function $f : X \to Y$ is said to be *semi-continuous* [4] if for each open set V of Y, $f^{-1}(V) \in \mathrm{SO}(X)$. We shall begin by giving a useful characterization of regular-closed functions.

THEOREM 2.1. *A function* $f : X \to Y$ *is regular-closed if and only if* $\mathrm{Cl}_Y(f(A)) \subset f(\mathrm{Cl}_X(A))$ *for each* $A \in \mathrm{SO}(X)$.

Proof. Necessity. Suppose that f is regular-closed and $A \in \mathrm{SO}(X)$. By Theorem 1 of [4], $A \subset \mathrm{Cl}_X(\mathrm{Int}_X(A))$ and $f(\mathrm{Cl}_X(\mathrm{Int}_X(A)))$ is closed. Therefore, we have $\mathrm{Cl}_Y(f(A)) \subset f(\mathrm{Cl}_X(A))$.

Sufficiency. Let A be a regular closed set of X. Then $A \in \mathrm{SO}(X)$ and, by hypothesis, $\mathrm{Cl}_Y(f(A)) \subset f(\mathrm{Cl}_X(A)) = f(A)$. Therefore, $f(A)$ is closed and hence f is regular-closed.

(*) Nella seduta del 21 novembre 1981.

The following theorem is an improvement of Theorem 2.20 of [5].

THEOREM 2.2. *Let* $f : X \to Y$ *be a regular-closed function. If* $A \in SO(X)$ *and there exists a* $B \subset Y$ *such that* $A = f^{-1}(B)$, *then* $f \mid A : A \to B$ *is regular-closed.*

Proof. Let $U \in SO(A)$. Since $A \in SO(X)$, by Theorem 1 of [6], $U \in SO(X)$ and hence $Cl_Y(f(U)) \subset f(Cl_X(U))$ by Theorem 2.1. Therefore, we have $Cl_B((f \mid A)(U)) = Cl_Y(f(U)) \cap B \subset f(Cl_X(U)) \cap B = f(Cl_X(U) \cap A) = f(Cl_A(U))$. Therefore, it follows from Theorem 2.1 that $f \mid A : A \to B$ is regular-closed.

COROLLARY 2.3. *If* $f : X \to Y$ *is regular-closed and semi-continuous, then for each open set* B *of* Y $f \mid A : A \to B$ *is regular-closed and semi-continuous, where* $A = f^{-1}(B)$.

Proof. Let B be open in Y. Since $A = f^{-1}(B) \in SO(X)$, it follows from Theorem 2.2 that $f \mid A : A \to B$ is regular-closed. Let V be an open set of B. Since B is open in Y, so is V and $f^{-1}(V) \in SO(X)$. By Theorem 6 of [4], $f^{-1}(V) = (f \mid A)^{-1}(V) \in SO(A)$. Therefore, $f \mid A$ is semi-continuous.

A function $f : X \to Y$ is said to be *almost-continuous* (*θ-continuous*) [8] if for each $x \in X$ and each open set V of Y containig $f(x)$, there exists an open set U of X containing x such that $f(U) \subset Int_Y(Cl_Y(V))$ (resp. $f(Cl_X(U)) \subset Cl_Y(V)$). It is well known that every almost-continuous function is θ-continuous but the converse is not true in general [3, 8]. In Theorem 2.18 of [5], it has been shown that a function of an H-closed space into a Hausdorff space is regular-closed if the graph function is almost-continuous and almost-open. This theorem may be considerably sharpened as follows:

THEOREM 2.4. *Let* $f : X \to Y$ *be a function of a quasi* H-*closed space* X *into a Hausdorff space* Y. *If the graph function* $g : X \to X \times Y$ *is* θ-*continuous, then* f *is regular-closed.*

Proof. The following results imply that f is regular-closed. A function $f : X \to Y$ is θ-continuous if and only if the graph function $g : X \to X \times Y$ is θ-continuous [7, Theorem 2]. Every θ-continuous function of a quasi H-closed space into a Hausdorff space is regular-closed [1, Theorem 3.2].

3. CLOSED FUNCTIONS

In this section we give some sufficient conditions for a regular-closed function to be closed. For a function $f : X \to Y$, the subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the *graph* of f and is denoted by $G(f)$. In [5, Theorem 2.22], it has been shown that if X is a normal space and $f : X \to Y$ is a regular-closed surjection with closed point inverses, then f is closed. The following theorem shows that the condition of f " *surjection with closed point inverses* " in the previous theorem can be dropted.

LEMMA 3.1 (Fuller [2]). *Let* $f : X \to Y$ *be a function with a closed graph. If* K *is a compact set of* X(Y), *then* $f(K)$ (*resp.* $f^{-1}(K)$) *is closed in* Y (*resp.* X).

THEOREM 3.2. *If* X *is a normal space and* $f : X \to Y$ *is a regular-closed function, then* f *is closed.*

Proof. Let A be a closed set of X and $y \in Y - f(A)$. First, we suppose that $y \notin f(X)$. Since X is regular closed in X, $f(X)$ is closed in Y. Put $V = Y - f(X)$, then V is an open set of Y such that $y \in V$ and $V \cap f(A) = \varnothing$. This implies that $y \notin \mathrm{Cl}_Y(f(A))$. Next, if $y \in f(X)$, then $f^{-1}(y) \neq \varnothing$ and $f^{-1}(y) \cap A = \varnothing$. Since f is regular-closed, by Corollary 2.14 of [5], G(f) is closed and $f^{-1}(y)$ is closed in X by Lemma 3.1. Since X is normal, there exist disjoint regular open sets U_y and U_A such that $f^{-1}(y) \subset U_y$ and $A \subset U_A$. By Lemma 2.2 of [5], there esists an open set V of Y such that $y \in V$ and $f^{-1}(V) \subset U_y$. Therefore, we have $V \cap f(A) = \varnothing$ and hence $y \notin \mathrm{Cl}_Y(f(A))$. Consequently, we obtain $\mathrm{Cl}_Y(f(A)) \subset f(A)$ for any closed set A of X. This shows that f is closed.

COROLLARY 3.3 (Long and Herrington [5]). *Normality is preserved under continuous regular-closed surjections.*

Proof. It is well known that normality is preserved under continuous closed surjections. The proof follows from Theorem 3.2.

THEOREM 3.4. *If* X *is a regular space and* $f : X \to Y$ *is a regular-closed functions with compact point inverses, then* f *is closed.*

Proof. The proof is similar to that of Theorem 3.2 and is thus omitted.

THEOREM 3.5. *If* X *is a compact space and* $f : X \to Y$ *is a regular-closed function, then* f *is closed.*

Proof. Since f is regular-closed, by Corollary 2.14 of [5], G(f) is closed in $X \times Y$. Let A be any closed set of X. Since X is compact, A is a compact set of X and hence $f(A)$ is closed in Y by Lemma 3.1. This shows that f is closed.

COROLLARY 3.6. *Let* X *be a compact space. Then, for a function* $f : X \to Y$ *the following properties are equivalent*:

(1) G(f) *is closed.*
(2) f *is closed.*
(3) *The graph function* $g : X \to X \times Y$ *is regular-closed.*
(4) f *is regular-closed.*

Proof. It follows from the proof of Theorem 3.5 that (1) implies (2). By Theorem 2.13 of [5], (2) implies (3) and (3) implies (4) by Theorem 2.16 of [5]. It follows from Corollary 2.14 of [5] that (4) implies (1).

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