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## $\mathcal{L}$-homomorphisms of Lie algebras

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**Algebra.** — *$\mathscr{L}$-homomorphisms of Lie algebras.* Nota di ALEXANDER A. LASHI, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si studiano gli omomorfismi reticolari ($\mathscr{L}$–omomorfismi) di algebre di Lie sopra anelli commutativi con unità. Le algebre di Lie sopra un campo e le $p$–algebre di Lie non ammettono $\mathscr{L}$–omomorfismi propri. Si assegnano condizioni necessarie e sufficienti affinchè un'algebra di Lie periodica o mista possieda un $\mathscr{L}$–omomorfismo su una catena di lunghezza $n$.

## INTRODUCTION

Let A be an universal algebra; $\mathscr{L}$ (A) denotes the lattice of all subalgebras of A.

DEFINITION. By *lattice homomorphism* or *$\mathscr{L}$-homomorphism* of an universal algebra A we mean a homomorphism $f$ of the lattice $\mathscr{L}$ (A) onto some lattice $\mathscr{L}$ ($f : \mathscr{L}(\mathrm{A}) \rightarrow \mathscr{L}$).

In the case when A is a group this problem is well known and was studied by M. Suzuki [5], G. Zappa [3], [4], [6], D. G. Higman [7], S. Sato [8], [9] and others.

The aim of this article is the study of $\mathscr{L}$-homomorphisms for Lie algebras. We will make use of generally accepted terminology and notations (see, for example, [1], [2], [10]).

Throughout this paper, the term " algebra " will always refer to a Lie algebra L over a commutative ring K which contains an identity and no zero divisors. These properties of K will not always be explicitly stated in what follows; moreover, additional conditions on K will be sometimes specified.

An element $l \in \mathrm{L}$ will be called *proper* if $kl \neq 0$ for every $k \in \mathrm{K}$ $(k \neq 0)$; otherwise, it will be called *periodic*; a Lie algebra L will be called *periodic* if all of its elements are periodic; L will be called *mixed* (or *nonperiodic*) if L contains both proper and periodic elements; and it will be called *proper* if all of its elements are proper.

The set of all periodic elements of L will be denoted by $t$ (L). It is clear that $t$ (L) is an ideal in L.

The dimension of L, denoted by dim L, is defined to be the maximal number of linearly independent elements. It is clear that $\dim (\mathrm{L}/t(\mathrm{L})) = \dim \mathrm{L}$.

*Notations.* $\mathrm{Z}_n$ is an $n$-dimension chain $(n \geq 1)$ ; $\mathrm{Z}_\infty = \cup_{k=1}^{\infty} \mathrm{Z}_k$ ; $\overline{\mathrm{Z}}$ is a free K-module, when $\dim \mathrm{Z} = 1$ ; alg (X) is the subalgebra generated by the set X.

(*) Nella seduta del 14 febbraio 1981.

A $\mathscr{L}$-homomorphism $f$ will be called *proper* if $f$ is neither a lattice isomorphism nor a trivial homomorphism [1].

## 1. LIE ALGEBRAS OVER A FIELD

Suppose now that L is a Lie algebra, not necessarily finite dimensional, over a field P.

LEMMA. *A two dimensional Lie algebra* L *over a field does not admit a* $\mathscr{L}$*-homomorphisms onto* $Z_2$.

*Proof.* Suppose that $\dim L = 2$ and $f : \mathscr{L}(L) \to Z_2 = (0, 1)$ is a $\mathscr{L}$-homomorphism. For each subalgebra $A_i \subseteq L$ we have

$$f(A_i) = \begin{cases} 0, & \text{if } A_i = 0; \\ 0, & \text{if } \dim A_i = 1 \quad \text{and} \quad i \in 1_1; \\ 1, & \text{if } \dim A_i = 1 \quad \text{and} \quad i \in 1_2; \\ 1, & \text{if } A_i = L. \end{cases}$$

One of the sets $1_1$ or $1_2$ is necessary it contains more than two elements. If $1_1$, then we have

$$f(A_i) = f(A_j) = 0 \Rightarrow 1 = f(L) = f(A_i \cup A_j) \neq f(A_i) \cup f(A_j) = 0.$$

If $1_2$ has more than two elements, then

$$f(A_i) = f(A_j) = 1 \Rightarrow 0 = f(A_i \cap A_j) \neq f(A_i) \cap f(A_j) = 1.$$

COROLLARY 1.1. *If a Lie algebra* L *over a field admits no trivial* $\mathscr{L}$*-homomorphism onto a chain* $Z_2$, *then* $\dim L = 1$.

COROLLARY 1.2. *Let* L *and* $L_1$ *be Lie algebras over a field* P *and* $f : \mathscr{L}(L) \to \mathscr{L}(L_1)$ *is* $\mathscr{L}$*-homomorphism, then* $f$ *is a lattice isomorphism.*

To make use of these facts, one may prove

THEOREM 1. *A Lie algebra* L *over a field admits no proper* $\mathscr{L}$*-homomorphism.*

In the rest of this note we always mean that K *is not a field* and study $\mathscr{L}$-homomorphisms of Lie algebras over K.

## 2. $\mathscr{L}$-HOMOMORPHISMS OF ALGEBRAS OVER RINGS

A Subalgebra $X \subset L$ is called *cyclic* if $X = \mathrm{alg}(x)$, for some $x \in L$. A Lie algebra L is called *locally cyclic* if each finite set of its elements generates a cyclic subalgebra. A proper Lie algebra L is locally cyclic if and only if $\dim L = 1$.

(1) We must not choose proper algebra with proper homomorphism.

DEFINITION. The $\mathscr{L}$-homomorphism $f : \mathscr{L}(L) \to \mathscr{L}$ will be called *full* if it satisfies the conditions

$$f(\cup_i L_i) = \cup_i f(L_i) \quad , \quad f(\cap_i L_i) = \cap_i f(L_i) ,$$

for every set of indexes $(i \in I)$.

The simple example shows that not each $\mathscr{L}$-homomorphism is full.

*Example.* Define the $\mathscr{L}$-homomorphism $f : \mathscr{L}(\overline{Z}) \to Z_2$ as follows

$$f(X_i) = \begin{cases} 0 , & \text{if } X_i = 0 , \\ 1 , & \text{if } 0 \subset X_i \subseteq Z . \end{cases}$$

It is not difficult to see, that the $\mathscr{L}$-homomorphism $f$ is not full.

PROPOSITION 2.1. *If* L *is a proper locally cyclic Lie algebra over a ring* K, *then there exists a full $\mathscr{L}$-homomorphism* $f : \mathscr{L}(L) \to \mathscr{L}(\overline{Z})$.

PROPOSITION 2.2. *Let* L *be a Lie algebra over a principal ideal domain and* $f : \mathscr{L}(L) \to \mathscr{L}(\overline{Z})$ *be a full $\mathscr{L}$-homomorphism, then* L *is a proper Lie algebra and* $\dim L = 1$.

So we have

THEOREM 2. *A Lie algebra* L *over a principal ideal domain* K *admits a full $\mathscr{L}$-homomorphism onto* $\mathscr{L}(\overline{Z})$ *if and only if* L *is a proper locally cyclic algebra.*

PROPOSITION 2.3. A mixed Lie algebra L admits an $\mathscr{L}$-homomorphism onto $Z_n$, when $\dim L = 1$ and admits an $\mathscr{L}$-homomorphism onto $Z_2$ if and only if, $\dim L = 1$.

## 3. $\mathscr{L}$-HOMOMORPHISMS OF PERIODIC AND MIXED LIE ALGEBRAS

In the rest of this note we always mean that K is a *principal ideal domain.*

DEFINITION. Let $p$ be a prime element of K; a periodical Lie algebra L over K will be called a Lie *p-algebra* if for each element $l \in L$ we have $\operatorname{Ann} l = p^n$ $(1 \leq n < \infty)$.

THEOREM 3. *A Lie p-algebra admits a proper $\mathscr{L}$-homomorphism if and only if it is locally cyclic.*

If L is a periodical Lie algebra over a principal ideal domain K, then L has expression in the direct product

$$(*) \qquad L = L_{p_1} + L_{p_2} + \cdots + L_{p_i} + \cdots$$

where $L_{p_i}$ are Lie $p_i$-algebras and $p_i \neq p_j$ $(i \neq j)$.

This decomposition will be called *natural* or N-*decomposition* of L.

An algebra L is said to be *$\mathscr{L}$-decomposable* if $\mathscr{L}(\mathrm{L})$ is decomposed into a direct product of two or more lattices none of which is one-element lattice.

An analogous to a M. Suzuki's theorem for Lie algebras holds.

LEMMA (Suzuki). *A Lie algebra* L *is $\mathscr{L}$-decomposable and its lattice of subalgebras* $\mathscr{L}(\mathrm{L}) = \Pi_i \mathscr{L}_i \, (i \in 1)$ *if and only if* L *is* N-*decomposable in the form* (*) *and* $\mathscr{L}(\mathrm{L}_i) = \mathscr{L}_i \; (i \in 1)$.

So it is clear that the study of $\mathscr{L}$-homomorphisms onto a periodical cyclic Lie algebra is the same as studying $\mathscr{L}$-homomorphisms onto the lattice

$$\mathscr{L} = \mathrm{Z}_{n_1} + \mathrm{Z}_{n_2} + \cdots + \mathrm{Z}_{n_k}.$$

COROLLARY 3.1. *If a periodical Lie algebra* L *is not a p-algebra, then* L *admits proper $\mathscr{L}$-homomorphisms.*

COROLLARY 3.2. *A periodical Lie algebra* L *admits proper $\mathscr{L}$-homomorphisms if and only if* L *is locally cyclic or is not a p-algebra.*

COROLLARY 3.3. A *periodical Lie algebra* L *admits proper $\mathscr{L}$-homomorphisms onto the lattice* $\mathscr{L} = \mathrm{Z}_{n_1} + \mathrm{Z}_{n_2} + \cdots + \mathrm{Z}_{n_k}$ *if and only if* L *is* N-*decomposable in the form*

$$\mathrm{L} = \mathrm{X}_{p_1} + \mathrm{X}_{p_2} + \cdots + \mathrm{X}_{p_m} + \mathrm{L}_{j_1} + \mathrm{L}_{j_2} + \cdots + \mathrm{L}_{j_t} + \cdots$$

*where* $\mathrm{X}_{p_i}$ *are locally cyclic $p_i$-algebras, Ann* $\mathrm{X}_{p_i} = p^{k_i} \; (k_i \leq \infty)\,,\, n \leq k_i$.

*Remark 1.* This question for groups was studied only in finite cases and the structure of those groups which admit $\mathscr{L}$-homomorphisms onto a finite cyclic group is rather difficult.

*Remark 2.* There is the possibility to find necessary and sufficient conditions for the $\mathscr{L}$-homomorphism of a periodical Lie algebra onto the lattice

$$\mathscr{L} = \mathrm{Z}_{n_1} + \mathrm{Z}_{n_2} + \cdots + \mathrm{Z}_{n_k} + \cdots + \mathrm{Z}_{\infty} + \cdots + \mathrm{Z}_{\infty} + \cdots.$$

DEFINITION. Let $f : \mathscr{L}(\mathrm{L}) \to \mathscr{L}$ be a $\mathscr{L}$-homomorphism and 0 the least element of $\mathscr{L}$, then the *lower kernel* $\ker(l, f, \mathrm{L}, \mathscr{L})$ is determined as following

$$\ker(l, f, \mathrm{L}, \mathscr{L}) = \cup_\lambda \mathrm{L}_\lambda \quad , \quad f(\mathrm{L}_\lambda) = 0\,, \qquad \lambda \in \Lambda\,.$$

If 1 is the greatest element of $\mathscr{L}$, then the *upper kernel* $\ker(u, f, \mathrm{L}, \mathscr{L})$ is determined as dyal

$$\ker(u, f, \mathrm{L}, \mathscr{L}) = \cap_\lambda \mathrm{L}_\lambda \quad , \quad f(\mathrm{L}_\lambda) = \mathrm{I}\,, \qquad \lambda \in \Lambda\,.$$

PROPOSITION 3.1. *ker* $(l, f, \mathrm{L}, \mathscr{L})\,\Delta\mathrm{L}$; *ker* $(u, f, \mathrm{L}, \mathscr{L})\,\Delta\mathrm{L}$.

DEFINITION. A $\mathscr{L}$-homomorphism $f : \mathscr{L}(\mathrm{L}) \to \mathscr{L}$ is called *regular* if $f(\ker(l, f, \mathrm{L}, \mathscr{L})) = 0$.

PROPOSITION 3.2. *If $f$ is not regular, then* $\ker(l, f, L, \mathscr{L}) = L$.

THEOREM 4. *A mixed Lie algebra* L *admits a regular $\mathscr{L}$-homomorphism onto* $Z_n$ *with ker* $(l, f, L, \mathscr{L})$ *if and only if* $\dim L = 1$ *and satisfies one of the following*

1) $t(L) \supseteq \ker(l, f, L, \mathscr{L})$ *and* $t(L)$ *admits an $\mathscr{L}$-homomorphism onto* $Z_{n-1}$

2) $t(L) \subseteq \ker(l, f, L, \mathscr{L})$, $L/\ker(l, f, L, \mathscr{L})$ *is a $p$-algebra of Lie and* $t(L)$ *has no $p$-elements.*

THEOREM 5. *A mixed Lie algebra* L *admits a regular $\mathscr{L}$-homomorphism onto the lattice* $\mathscr{L} = \underbrace{Z_2 + Z_2 + \cdots + Z_2}_{k}$ *if and only if* $\dim L = 1$ *and for some prime elements* $p_1, p_2, \cdots, p_k \in K$ *the subalgebra* $t(L)$ *has no $p_i$-elements.*

Remark, that an element $l \in L$ is called a *$p$-element* if alg $(l)$ is a $p$-algebra of Lie.

The theorems 2, 4, 5 and corollaries 3.2, 3.3 contain all analogues to theoretical group results (see [1], [3]-[9]).

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