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On the Holomorphic Endomorphisms of the Ball

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Geometria. — *On the Holomorphic Endomorphisms of the Ball.* Nota di GIOVANNI BASSANELLI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Sia F un endomorfismo olomorfo della palla unitaria aperta B_n di $\mathbf{C}^n$. In questa nota proviamo che certe ipotesi minimali, relative al comportamento di F su un orociclo e vicino ad un punto del bordo, assicurano che F è un automorfismo olomorfo di B_n.

INTRODUCTION

It is well known that the metrics of Kobayashi, Carathéodory and Bergman on the open unit ball B_n for the euclidean norm of $\mathbf{C}^n$ are coincident. We shall make reference to any of these metrics and we shall denote with d the associate distance.

In this paper we consider a holomorphic endomorphism F of B_n, and we show that, if F behaves " regularly " on a horocycle and close to a boundary point of B_n, then $F \in \operatorname{Aut}(B_n)$ (cf. Theorem II).

1. Let W be a point of the boundary of B_n. The horosphere tangent to ∂B_n at W of index $k > 0$ is the set

$$H(k, W) = \{Z \in \mathbf{C}^n ; |1 - \langle Z, W\rangle|^2 < k(1 - \|Z\|^2)\}.$$

The exhorosphere tangent to ∂B_n at W of index $h > 0$ is the set

$$E(h, W) = \{Z \in \mathbf{C}^n ; |1 - \langle Z, W\rangle|^2 > h(\|Z\|^2 - 1)\}.$$

The boundaries of these regions are called, respectively, horocycles and exhorocycles.

For any $A \in B_n$, setting $a(A) = \sqrt{1 - \|A\|^2}$, the map

$$T_A(Z) = \frac{\langle Z, A\rangle}{1 + a(A)} A + a(A) Z, \qquad Z \in \mathbf{C}^n$$

is a linear isomorphism of $\mathbf{C}^n$, and the function

$$Z \mapsto f_A(Z) = T_A\left(\frac{Z - A}{1 - \langle Z, A\rangle}\right)$$

defines a biholomorphic map $\{Z \in \mathbf{C}^n ; \langle Z, A\rangle \neq 1\} \to \{Z \in \mathbf{C}^n ; \langle Z, A\rangle \neq -1\}$. Moreover $\operatorname{Aut}(B_n) = \{(U \circ f_A)|_{B_n} ; A \in B_n$ and U is an unitary trasforma-

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tion of C^n}. Note, in particular, that every automorphism of B_n is defined in an open neighbourhood of $\overline{B}_n$. (For all these facts, cf. [1]).

Let $Z_1, Z_2 \in \mathbf{C}^n$ be such that $\langle Z_j, A\rangle \neq 1$ $(j = 1, 2)$. Put $Z_j = q_j V + Q_j$ with $V = \frac{A}{\|A\|}$, $q_j = \langle Z_j, V\rangle$, $Q_j = Z_j - \langle Z_j, V\rangle V$, $j = 1, 2$. Then a trivial computation yields

$$1 - \langle f_A(Z_1), f_A(Z_2)\rangle = \frac{(1 - \|A\|^2)(1 - \langle Z_1, Z_2\rangle)}{(1 - \langle Z_1, A\rangle)(1 - \langle A, Z_2\rangle)},$$

and this equality implies readily

LEMMA 1. *Let* F *be a holomorphic automorphism of* B_n, *and for* $W \in \partial B_n$, *let*

$$r = \frac{1 - \|F^{-1}(0)\|^2}{|1 - \langle W, F^{-1}(0)\rangle|^2}.$$

Then

(i) $F(\partial H(k, W)) = \partial H(rk, F(W))$, $\quad k > 0$;

(ii) *If* F *is defined in* Z *and* $h > 0$, *then*

$$Z \in \partial E(h, W) \Longleftrightarrow F(Z) \in \partial E(rh, F(W)).$$

LEMMA 2. *Let* $\tilde{H}_1$ *be a horocycle of* B_n *tangent to* ∂B_n *at* $W \in \partial B_n$, *and let* $V \in \partial B_n \setminus \{W\}$. *There exists* $F \in \mathrm{Aut}(B_n)$ *such that*

(i) $F(\tilde{H}_1)$ *is the horocycle* $\tilde{H} = \partial H(1, e_1)$ *passing through the point* 0 *and tangent ot* ∂B_n *at* $e_1 = (1, 0, \cdots, 0)$;

(ii) $F(V) = -e_1$.

Proof. Let g be the geodesic from W to V, $A \in g \cap \tilde{H}_1 \setminus \{W\}$, and $F = (U \circ f_A)|_{B_n}$ with U unitary such that $U(f_A(V)) = -e_1$.

Since $F(g)$ is a geodesic through $F(A) = 0$, then $F(g)$ is a diameter whose extremes are $F(V) = -e_1$ and $F(W)$. Therefore $F(\tilde{H}_1)$ is the horocycle passing through $F(A) = 0$ and tangent to ∂B_n at $F(W) = e_1$.

Q.E.D.

2. To prove our result, we have to distinguish the case $n = 1$ from the others.

THEOREM I. *Let* $F : B = B_1 \to B$ *be a holomorphic function. Let* $\tilde{H}_1, \tilde{H}_2$ *be horocycles tangent to* ∂B *at* $W_1, W_2 \in \partial B$ *respectively. Suppose that*:

(i) $F^{-1}(\tilde{H}_2 \setminus \{W_2\}) \cap \tilde{H}_1$ *is not empty and has at least an accumulation point distinct from* W_1;

(ii) *There exist* $V_1, V_2 \in \partial B$, $V_1 \neq W_1$, *and a sequence* $(Z_\nu)_{\nu \in N}$ *in* B *such that* $\lim_{\nu \to \infty} Z_\nu = V_1$ *and* $\lim_{\nu \to \infty} F(Z_\nu) = V_2$.

Then $F \in \mathrm{Aut}(B)$.

Proof. By Lemma 2 we can assume $\tilde{H}_1 = \tilde{H}_2 = \tilde{H}$, $W_1 = W_2 = 1$, $V_1 = -1$.

Let $a(t) = \dfrac{1+t}{2}$, $t \in \mathbf{C}$. Then $D = a^{-1}(B)$ is the open ball with center -1 and radius 2. Let $G = a^{-1} \circ F \circ a : D \to D$. The function

$$L(t) = \frac{1}{\overline{G(1/\bar{t})}}$$

is defined and holomorphic when $1/\bar{t} \in D$ and $G(1/\bar{t}) \neq 0$.

Let $t \in G^{-1}(\partial B \setminus \{1\}) \cap \partial B$. Since $1/\bar{t} = t \in D$ and $0 \neq G(1/t) = G(t) \in \partial B$, then $L(t) = G(t)$. Since, by (*i*), the set $G^{-1}(\partial B \setminus \{1\}) \cap \partial B = a^{-1}(F^{-1}(\tilde{H} \setminus \{1\}) \cap \tilde{H})$ has some accumulation point in $\partial B \setminus \{1\}$ $(1 = a^{-1}(1) = a^{-1}(W_1))$, then L is the analytic extension of G. It follows that

$$G(t) = \frac{1}{\overline{G(1/\bar{t})}} \tag{1}$$

whenever $1/\bar{t} \in D$ and $G(1/\bar{t}) \neq 0$.

Let $s \in \partial B \setminus \{1\}$. Since $1/\bar{s} = s \in D$, we can assume that (1) holds for every t sufficiently close to s, $t \neq s$. Then

$$G(s) = \lim_{t \to s} G(t) = \lim_{t \to s} \frac{1}{\overline{G(1/\bar{t})}}. \tag{2}$$

Since s is not a pole for G, $G(1/\bar{s}) \neq 0$. Therefore, by (2), (1) holds for $t = s$, i.e. $G(s) = \dfrac{1}{\overline{G(1/\bar{s})}} = \dfrac{1}{\overline{G(s)}}$. Hence $G(s) \in \partial B \setminus \{1\}$ and in conclusion $G(\partial B \setminus \{1\}) \subset \partial B \setminus \{1\}$, i.e.

$$F(\tilde{H} \setminus \{1\}) \subset \tilde{H} \setminus \{1\}. \tag{3}$$

Since $-1/3 \in D$ and $-3 \in \partial D$ is not a pole for G, by the same argument we can prove that $G(-3) = \dfrac{1}{\overline{G(-1/3)}}$. By (*ii*)

$$G(-3) = G(a^{-1}(V_1)) = (G \circ a^{-1})(\lim_{\nu \to \infty} Z_\nu) = \lim_{\nu \to \infty} G \circ a^{-1}(Z_\nu) =$$

$$= \lim_{\nu \to \infty} a^{-1} F(Z_\nu) = a^{-1}(V_2).$$

If $V_2 = 1$, then $1 = \dfrac{1}{\overline{a^{-1}(V_1)}} = G(-1/3) \in D$. But this is a contradiction. Thus $V_2 \neq 1 = W_2$, and in view of Lemma 2 we can assume $V_2 = -1$. It follows that $G(-1/3) = -1/3$, i.e.

$$F(1/3) = 1/3. \tag{4}$$

In view of (3), $F(0) \in \tilde{H} \setminus \{1\}$. Since F is a contraction for the distance d then, by (4), $d(0, 1/3) \geq d(F(0), 1/3)$. On the other hand $d(Z, 1/3) > d(0, 1/3)$ whenever $0 \neq Z \in \tilde{H} \setminus \{1\}$. Thus $F(0) = 0$.

Now the theorem follows from the Schwarz lemma. Q.E.D.

Remark 1. Hypothesis (*ii*) cannot be dropped. In fact the function $F(Z) = \dfrac{Z}{3 - 2Z}$, $Z \in B$, is a biholomorphic automorphism of $H(1, 1)$ such that $F(B) \subset B$, but $-1/2 \in B \setminus F(B)$.

The following has been established by H. Alexander [2];

LEMMA 3. *Let* S *be a domain of* C^n, $n > 1$, *such that* $S \cap \partial B_n \neq \varnothing$. *Let* $T: S \to C^n$ *be a holomorphic map such that* $T(S \cap \partial B_n \subset \partial B_n$. *Then either* T *is a constant map or* T *extends to be a holomorphic autormorphism of* B_n.

THEOREM II. *Let* $F: B_n \to B_n$ *be a holomorphic map. Let* $\tilde{H}_1, \tilde{H}_2$ *be horocycles.*

If the following two conditions are both satisfied

(i) *There exists an open subset* S *of* B_n *such that*

$$S \cap \tilde{H}_1 = \varnothing \qquad \textit{and} \quad F(S \cap \tilde{H}_1) \subset \tilde{H}_2;$$

(ii) *There exist* $V_1, V_2 \in \partial B_n$, $V_1 \notin \tilde{H}_1$ *and a sequence* $(Z_\nu)_{\nu \in N}$ *in* B_n *such that* $\lim_{\nu \to \infty} Z_\nu = V_1$ *and* $\lim_{\nu \to \infty} F(Z_\nu) = V_2$,

then
$$F \in \mathrm{Aut}(B_n).$$

Proof. If $n = 1$ this is Theorem I. So we assume $n > 1$. By Lemma 2 there is no restriction in assuming $\tilde{H}_1 = \tilde{H}_2 = \tilde{H}$. For $Z = (z_1, Z')$ with $z_1 \in \mathbf{C}$, $Z' \in \mathbf{C}^{n-1}$, let

$$b(Z) = \left(\frac{z_1 + 1}{2}, \frac{1}{\sqrt{2}} Z'\right).$$

Then $b^{-1}(B_n) = E(2, e_1)$, $b^{-1}(\tilde{H}) = \partial B_n$. Let $G = b^{-1} \circ F \circ b : E(2, e_1) \to E(2, e_1)$. By (*i*) $G(b^{-1}(S) \cap \partial B_n) \subset \partial B_n$ and $b^{-1}(S)$ is an open subset of $\mathbf{C}^n$ such that $b^{-1}(S) \cap \partial B_n \neq \varnothing$.

Since $\lim_{\nu \to \infty} G(b^{-1}(Z_\nu)) = b^{-1}(V_2) \in \partial E(2, e_1)$ with $b^{-1}(Z_\nu) \in E(2, e_1)$, G is not a constant map. Lemma 3 implies that $G|_{B_n} \in \mathrm{Aut}(B_n)$. Hence there exist a vector $A \in B_n$ and an unitary operator U such that, whenever $\langle Z, A \rangle \neq 1$.

$$G(Z) = (U \circ T_A)\left(\frac{Z - A}{\langle 1 - Z, A \rangle}\right).$$

Since G is bounded in $E(2, e_1)$ and $b^{-1}(V_1) \in \partial E(2, e_1)$, $\langle b^{-1}(V_1), A \rangle \neq 1$. It follows that G is holomorphic in an open neighbourhood S_1 of $b^{-1}(V_1)$, and $G(b^{-1}(V_1)) = b^{-1}(V_2)$.

Since G is bijective on ∂B_n and since $e_1 \in E(2, e_1) \supset G(\partial B_n \setminus \{e_1\})$, then $G(e_1) = e_1$. Being G injective, then $b^{-1}(V_2) = G(b^{-1}(V_1)) \neq G(e_1) = e_1$. By Lemma 1 there exists $r > 0$ such that

$$W \in S_1 \cap \partial E(2, e_1) \Rightarrow G(W) \in \partial E(2r, G(e_1)) = \partial E(2r, e_1).$$

In particular $b^{-1}(V_2) = G(b^{-1}(V_1)) \in \partial E(2r, e_1) \cap \partial E(2, e_1) \setminus \{e_1\}$. Thus $r = 1$, and therefore $F(b(S_1) \cap \partial B_n) \subset \partial B_n$. The conclusion follows now from Lemma 3. Q.E.D.

Remark 2. The questions discussed in this paper make sense also in the infinite dimensional case. Some of the machinary involved in the proofs can be adapted also to the infinite dimensional case (e.g. the explicit description of the group Aut (B_∞) [3]). Unfortunately no proof of the crucial Lemma 3 is available in the infinite dimensional case.

References

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