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On the solutions of the inhomogeneous evolution equation in Banach spaces

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Analisi matematica. — *On the solutions of the inhomogeneous evolution equation in Banach spaces.* Nota di EUGENIO SINESTRARI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Vengono dati nuovi teoremi di regolarità per le soluzioni dell'equazione $u'(t) = \Lambda u(t) + f(t)$ nel caso in cui Λ è il generatore infinitesimale di un semigruppo analitico in uno spazio di Banach E e f è una funzione continua.

1. INTRODUCTION

Let E be a Banach space with norm $\|\cdot\|$ and $\Lambda : D_\Lambda \subset E \to E$ be the infinitesimal generator of a holomorphic semigroup $e^{\Lambda t}$. For each $k = 0, 1, \cdots$ let $M_k > 0$ be a constant such that for each $t > 0$ we have $\| t^k \Lambda^k e^{\Lambda t} \| \leq M_k$. We will study the equation

$$\text{(P)} \qquad \begin{cases} u'(t) = \Lambda u(t) + f(t) \\ u(0) = x \end{cases} \qquad 0 \leq t \leq T$$

for each given $f \in C(0, T; E)$ and $x \in E$.

Some conditions on f and x to obtain solutions of (P) are well known (see references): in this paper we want to add new results about the properties of the mild solutions and the maximal regularity of the classical solutions. These results will be applied to the study of the semilinear evolution equations ([10]).

2. DEFINITIONS AND PRELIMINARIES

Let us denote by $X = C(0, T; E)$ the Banach space of the continuous functions $u : [0, T] \to E$ with the sup-norm. We shall define the multiplication operator by Λ in X as follows:

DEFINITION 1. $A : D_A \subset X \to X$ where $D_A = C(0, T; D_\Lambda)$ (D_Λ is equipped with the graph norm) and $(Au)(t) = \Lambda u(t)$.

It is easy to prove that A is the generator of a holomorphic semigroup e^{At} in X such that $(e^{At} u)(s) = e^{\Lambda t} u(s)$ for $t \geq 0$ and $s \in [0, T]$.

DEFINITION 2. For each $f \in X$ we define $e^A * f$ as follows:

$$(1) \qquad (e^A * f)(t) = \int_0^t e^{\Lambda s} f(t-s) \, ds \qquad 0 \leq t \leq T.$$

$t \to e^{\Lambda t} x + (e^A * f)(t)$ is called the *mild solution* of (P).

(*) Work done as a member of GNAFA of CNR.
(**) Nella seduta del 16 gennaio 1981.

DEFINITION 3. Given $f \in X$, a function $u \in C^1(0, T; E) \cap C(0, T; D_A)$ is called a *solution* of (P) if (P) is verified for each $t \in [0, T]$.

It is known that if u is a solution of (P) then we must have

$$u(t) = e^{At} x + \int_0^t e^{As} f(t-s)\, ds . \tag{2}$$

DEFINITION 4. $C^\theta(0, T; E)$ is the space of *hölder continuous functions* $u : [0, T] \to E$ with exponent $\theta \in]0, 1[$ and norm

$$\|u\|_{C^\theta} = \|u\|_C + \sup_{t \neq s} \frac{\|u(t) - u(s)\|}{|t-s|^\theta} . \tag{3}$$

We set $C_0^\theta(0, T; E) = \{u \in C^\theta(0, T; E) ; u(0) = 0\}$.

DEFINITION 5. $h^\theta(0, T; E)$ is the space of *little hölder continuous functions* i.e.

$$h^\theta(0, T; E) = \left\{ u \in C^\theta(0, T; E) ; \lim_{\delta \to 0} \sup_{|t-s| \leq \delta} \frac{\|u(t) - u(s)\|}{|t-s|^\theta} = 0 \right\} ;$$

h^θ is the completion of C^1 in C^θ (see [3], [9]).

3. THE HOMOGENEOUS CASE

In this section we shall consider the following special case of (P)

$$\begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = 0 \end{cases} \qquad 0 \leq t \leq T \tag{P_0}$$

and we will also suppose that $f(0) = 0$.

The following theorem is well known (see [2], [4], [6], [11]).

THEOREM 1. *If* $f \in X$ *then* $u = e^A * f \in C_0^\theta$ *for each* $\theta \in]0, 1[$ *and*

$$\|u\|_{C^\theta} \leq c_1 \|f\|_X \tag{4}$$

THEOREM 2. *If* $f \in C_0^\theta$ *then* $u = e^A * f$ *is a solution of* (P_0) *and* u', $Au \in C_0^\theta$. *Moreover*

$$\|Au\|_{C^\theta} \leq c_2 \|f\|_{C^\theta} . \tag{5}$$

where $c_2 = c_2(T, \theta, M_0, M_1)$.

Proof. This result was proved in [2] as a consequence of an abstract theory about the sum of two operators. By direct methods, Kato [5] proved that when $f \in C^\theta$, $u' \in C^\theta(\varepsilon, T; E)$ for each $\varepsilon \in]0, T[$; we will see that the same elementary methods prove that: $u' \in C_0^\theta(0, T; E)$ if $f(0) = 0$.

Proceeding as in [5] we prove that $\Lambda u(t) = \Lambda u_1(t) + e^{\Lambda t} f(t) - f(t)$ and $\mathrm{A} u_1 \in \mathrm{C}_0^\theta(0, \mathrm{T}; \mathrm{E})$. When $0 < s < t$ we have

$$\|e^{\Lambda t} f(t) - e^{\Lambda s} f(s)\| \leq \|e^{\Lambda t}[f(t) - f(s)]\| + \|(e^{\Lambda t} - e^{\Lambda s}) f(s)\| \leq$$

$$\leq \mathrm{M}_0 \|f\|_{\mathrm{C}^\theta}(t-s)^\theta + \left\| \int_s^t \Lambda e^{\Lambda r} \, \mathrm{d}r \right\|_{\mathscr{L}(\mathrm{E})} \|f\|_{\mathrm{C}^\theta} s^\theta \leq \|f\|_{\mathrm{C}^\theta} \Big(\mathrm{M}_0 (t-s)^\theta +$$

$$+ \mathrm{M}_1 \int_s^t r^{\theta-1} \, \mathrm{d}r \Big) \leq (\mathrm{M}_0 + \mathrm{M}_1 \theta^{-1}) \|f\|_\theta (t-s)^\theta$$

hence $t \to e^{\Lambda t} f(t)$ is hölder continuous and the conclusion follows.

THEOREM 3. *Let* $f \in h_0^\theta$: *then* $u = e^{\mathrm{A}} * f$ *is a solution of* (P_0) *and* u', $\mathrm{A}u \in h_0^\theta$.

Proof. The first part follows from Theorem 2. Now let $f_n \in \mathrm{C}_0^1(0, \mathrm{T}; \mathrm{E})$ be such that $\lim_{n\to\infty} \|f_n - f\|_{\mathrm{C}^\theta} = 0$; setting $u_n = e^{\mathrm{A}} * f_n$, we deduce from (5) that $\|\mathrm{A}u_n - \mathrm{A}u\|_{\mathrm{C}^\theta} \leq c_2 \|f_n - f\|_{\mathrm{C}^\theta}$; as $\mathrm{A}u_n \in \mathrm{C}_0^\alpha$ for each $\alpha \in]0, 1[$ we have $\mathrm{A}u_n \in h_0^\theta$. From $\lim_{n\to\infty} \mathrm{A}u_n = \mathrm{A}u$ (in C^θ) we get $\mathrm{A}u \in h_0^\theta$.

Let us introduce now a family of intermediate spaces between D_Λ and E (for details and proofs see [1]).

DEFINITION 6. Let $\theta \in]0, 1[\cdot \mathrm{D}_\Lambda(\theta) = \{x \in \mathrm{E}, \lim_{t\to 0} t^{1-\theta} \Lambda e^{\Lambda t} x = 0\}$. $\mathrm{D}_\Lambda(\theta)$ is a Banach space with the norm $\|x\|_\theta = \|x\| + \sup_{t>0} \|t^{1-\theta} \Lambda e^{\Lambda t} x\|$. It can be proved that D_Λ is dense in $\mathrm{D}_\Lambda(\theta)$.

The following proposition gives an useful characterization of $\mathrm{D}_\mathrm{A}(\theta)$.

PROPOSITION 1. *Let* A *be given by Definition 1 and* $\theta \in]0, 1[$; *then* $\mathrm{D}_\mathrm{A}(\theta) = \mathrm{C}(0, \mathrm{T}; \mathrm{D}_\Lambda(\theta))$ *with equivalent norms. We shall set* $\mathrm{X}_\theta = \mathrm{C}(0, \mathrm{T}; \mathrm{D}_\Lambda(\theta))$.

Proof. When $u \in \mathrm{D}_\mathrm{A}(\theta)$ we have

$$\sup_{0 \leq t \leq \mathrm{T}} \|u(t)\|_\theta \leq \|u\|_{\mathrm{D}_\mathrm{A}(\theta)}. \tag{6}$$

There is $u_n \in \mathrm{D}_\mathrm{A}$ such that $\lim_{n\to\infty} u_n = u$ in $\mathrm{D}_\mathrm{A}(\theta)$ hence also in X_θ in consequence of (6).

Conversely if $u \in \mathrm{X}_\theta$ there is $u_n \in \mathrm{D}_\mathrm{A}$ such that $\lim_{n\to\infty} u_n = u$ in X_θ; as we have $\|u_n\|_{\mathrm{D}_\mathrm{A}(\theta)} \leq 2 \|u_n\|_{\mathrm{X}_\theta}$ we deduce that u_n converges in $\mathrm{D}_\mathrm{A}(\theta)$ to u.

THEOREM 4. *If* $f \in \mathrm{X}$ *then* $u = e^{\mathrm{A}} * f \in \mathrm{X}_\theta$ *for each* $\theta \in]0, 1[$ *and*

$$\|u\|_{\mathrm{X}_\theta} \leq c_3 \|f\|_{\mathrm{X}}. \tag{7}$$

Proof. For $\xi > 0$ and $0 \le t \le T$,

$$\| \xi^{1-\theta} \Lambda e^{\Lambda \xi} u(t) \| = \left\| \xi^{1-\theta} \int_0^t \Lambda e^{(s+\xi)\Lambda} f(t-s)\, ds \right\| \le$$

$$\le M_1 \|f\|_X \int_0^t \frac{\xi^{1-\theta}}{s+\xi}\, ds \le M_1 \|f\|_X \xi^{1-\theta} \lg \frac{T+\xi}{\xi},$$

which tends to zero as $\xi \to 0$; from Proposition 1 we deduce $u \in X_\theta$. By means of the inequality

$$\sup_{\xi > 0} \frac{\xi^{1-\theta}}{s+\xi} \le \frac{1}{s^\theta} \tag{8}$$

we obtain $\sup_{\xi>0} \| \xi^{1-\theta} \Lambda e^{\Lambda\xi} u(t) \| \le \frac{M_1}{1-\theta} T^{1-\theta} \|f\|_X$, and the conclusion follows.

THEOREM 5. *If* $f \in X$ *and* $0 < \alpha < 1 - \theta < 1$ *then* $u = e^A * f \in C_0^\alpha(0, T; D_\Lambda(\theta))$. *Moreover*

$$\|u\|_{C_0^\alpha(0,T;D_\Lambda(\theta))} \le c_4 \|f\|_X. \tag{9}$$

Proof. For $\xi > 0$, $0 \le t_1 < t_2 \le T$ we have $\| \xi^{1-\theta} \Lambda e^{\Lambda\xi} (u(t_2) - u(t_1)) \| =$ $= \left\| \xi^{1-\theta} \int_0^{t_1} \Lambda [e^{\Lambda(t_2-s+\xi)} - e^{\Lambda(t_1-s+\xi)}] f(s)\, ds + \xi^{1-\theta} \int_{t_1}^{t_2} \Lambda e^{\Lambda(t_2-s+\xi)} f(s)\, ds \right\|$. Now for each $\alpha \in]0,1[$ we have $\xi^{1-\theta} \| \Lambda e^{\Lambda(t_2-s+\xi)} - \Lambda e^{\Lambda(t_1-s+\xi)} \| \le M_2 \xi^{1-\theta} \int_{t_1-s+\xi}^{t_2-s+\xi} r^{-2}\, dr =$ $= M_2 (t_2 - t_1) \frac{\xi^{(1-\theta)\alpha}}{t_1 - s + \xi} \frac{\xi^{(1-\theta)(1-\alpha)}}{t_2 - s + \xi} \le M_2 (t_2 - t_1)(t_1 - s)^{(1-\theta)\alpha - 1} (t_2 - s)^{(1-\theta)(1-\alpha)-1}$ by using (8). Hence

$$\sup_{\xi>0} \| \xi^{1-\theta} \Lambda e^{\Lambda\xi} (u(t_2) - u(t_1)) \| \le M_2 \|f\|_X (t_2 - t_1)^{(1-\theta)(1-\alpha)} \int_0^{t_1} (t_1 - s)^{(1-\theta)\alpha - 1}\, ds +$$

$$+ M_1 \|f\|_X \int_{t_1}^{t_2} (t_2 - s)^{-\theta}\, ds \le \text{const.} \|f\|_X (t_2 - t_1)^{(1-\theta)(1-\alpha)}.$$

By using also Theorem 1 the conclusion follows.

The following theorem is proved in [3]:

THEOREM 6. *If* $f \in X_\theta$ *then* $u = e^A * f$ *is a solution of* (P_0) *and* u', $Au \in X_\theta$. *Moreover*

$$\|Au\|_{X_\theta} \le c_5 \|f\|_{X_\theta}. \tag{10}$$

4. The non-homogeneous case

Let us consider now problem (P) with $f \in X$ and $x \in E$. The mild solution of (P) is given by

$$u(t) = u_0(t) + \int_0^t e^{\Lambda s} f(0)\,ds + e^{\Lambda t} x \tag{12}$$

where u_0 is the mild solution of problem (P_0) with $f(t)$ replaced by $f(t) - f(0)$.

Definition 7. $D_\Lambda(\theta, \infty) = \{x \in E \,,\ \sup_{t>0} \|t^{1-\theta} \Lambda e^{\Lambda t} x\| < \infty\}$, where $0 < \theta < 1$.

Proposition 2. The function $t \to e^{\Lambda t} x$ is in $C^\theta(0, T; E)$ if $x \in D_\Lambda(\theta, \infty)$ and is in $h^\theta(0, T; E)$ if $x \in D_\Lambda(\theta)$.

The *proof* is a consequence of 2.3.6, 3.4.12 and 3.5.8 of [1].

By applying the preceding proposition we can get several results from the theorem of section 3. We shall consider only three cases. More results are given in [10].

Theorem 7.

(*i*) If $f \in C^\theta$, $x \in D_\Lambda$ and $\Lambda x + f(0) \in D_\Lambda(\theta, \infty)$ then (P) has a solution u such that u', $Au \in C^\theta$.

(*ii*) If $f \in h^\theta$, $x \in D_\Lambda$ and $\Lambda x + f(0) \in D_\Lambda(\theta)$ then (P) has a solution u such that u', $Au \in h^\theta$.

(*iii*) If $f \in C(0, T; D_\Lambda(\theta))$, $x \in D_\Lambda$ and $\Lambda x \in D_\Lambda(\theta)$ then (P) has a solution u such that u', $Au \in C(0, T; D_\Lambda(\theta))$.

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