Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Projective automorphisms of convex cones

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **69** (1980), n.6, p. 346–150.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1980_8_69_6_346_0>

Geometria. — *Projective automorphisms of convex cones.* Nota di GRAZIANO GENTILI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studia la struttura del gruppo degli automorfismi proiettivi di un cono aperto, regolare e convesso di uno spazio vettoriale reale. Si discute l'estendibilità degli elementi di questo gruppo ad automorfismi olomorfi del dominio tubolare associato al cono, ed il loro comportamento rispetto alla metrica riemanniana canonica del cono stesso.

Let R be an n-dimensional, real vector space, and let R′ be its dual; both of them endowed with their natural topology. We shall follow [1] for terminology and notations. If V is an open, convex, regular cone in R, then V′ is open, convex and regular in R′. By means of the *characteristic* function Φ_V of V (see [1], [5]) a (positive definite) Riemannian metric of class C^∞ can be defined on V, which turns out to be invariant under the action of the group GL (V) of all affine (hence linear, see [5]) automorphisms of V. This metric will be called henceforth the *canonical* Riemannian metric of V.

Let now $T(V) = \{r + iv : r \in R, v \in V\} \subset R + iR$ be the tube domain corresponding to the cone V. Each element of GL (V) can be "extended" to a holomorphic automorphism of T (V), in the sense that, for each $g \in GL(V)$, there exists a (unique) holomorphic automorphism G of T (V), such that $G(iv) = ig(v)$, for all $v \in V$ (see [2], [3]).

Let us consider the vector space $\hat{R} = R \times \mathbf{R}$, and let us define the real projective space $\mathbf{P}(\hat{R}) = \mathbf{P}$ of dimension n; from now on, R will always be considered as imbedded in $\mathbf{P}$ and identified with (the image of) the set $R \times \{1\} \subset \mathbf{P}$.

The group GL $(\hat{R})$ of linear isomorphisms of $\hat{R}$, induces on the quotient space $\mathbf{P}$ a group of transformations, which will be called the group of *projective* transformations, and denoted by GL ($\mathbf{P}$). The imbedding $R \hookrightarrow \mathbf{P}$, yields an injective homomorphism $GL(R) \hookrightarrow GL(\mathbf{P})$. In the following the same symbol will be used to indicate, either an element of GL (R), or its image in GL ($\mathbf{P}$).

At this point, the problem of studying the subgroup of GL ($\mathbf{P}$) consisting of all elements which transform the cone $V \subset R \hookrightarrow \mathbf{P}$ onto itself, arises naturally; this subgroup will be denoted by GL ($\mathbf{P}$, V), and its elements will be called *projective automorphisms* of V.

In this paper a characterization for all open, convex, regular cones of R for which the group GL ($\mathbf{P}$, V) strictly contains the group GL (V) will be given, and the structure of GL ($\mathbf{P}$, V) and its relationship with the "decomposition" of V into its "irreducible" parts, will be described.

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(**) Nella seduta del 6 dicembre 1980.

It turns out that, if $\dim R > 1$, then the elements of $GL(\mathbf{P}, V) \setminus GL(V)$ are not isometries for the canonical Riemannian metric of V. Furthermore they cannot be extended to holomorphic automorphisms of the tube domain T (V).

Detailed proofs and further results will appear elsewhere.

1. *Decompositions*. Two different "decompositions" for a cone $V \subset R$ can be defined.

DEFINITION 1.1. *Let* V *be a cone of the vector space* R. *Let* $R_1, R_2, \cdots, R_K$ *be vector subspaces of* R *with strictly positive dimensions, such that* $R_1 \oplus R_2 \oplus \cdots \oplus R_K = R$.

Let $W_1 \subset R_1$, $W_2 \subset R_2, \cdots, W_K \subset R_K$ *be cones, all different from* $\{0\}$.

The set $S = \{W_1, W_2, \cdots, W_K\}$ *of cones is called a* decomposition *of* V, *if*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_K .$$

A cone V *is called* irreducible, *if the only decomposition of* V *is the one for which* $k = 1$, $V = W_1$.

A decomposition S *of* V *is called* maximal, *if all its elements are irreducible.*

The following theorem holds, which was announced in a weaker form and without proof by E. B. Vinberg in [5]. (A proof in the case in which V is affinely homogeneous appears in [4]).

THEOREM 1.2. *If* V *is a convex, regular cone of* R, *then there exists one, and only one, maximal decomposition of* V.

From theorem 1.2 it follows:

COROLLARY 1.3. *If* V *is a convex, regular cone of* R, *then there exists one, and only one, decomposition* S *of* V *with the following properties*:

i) *At most one element of* S *is not a half-line.*

ii) *If one element* W_i *of* S *is different from a half-line, no decomposition of* W_i *contains a half-line.*

2. *Projective automorphisms*. From now on V will be an open, convex, regular cone of R. The same notation will indicate the image of V by the mapping $R \hookrightarrow \mathbf{P}$.

Let $\hat{\phi} \in GL(\hat{R})$ be defined by

$$\hat{\phi}(r, \alpha) = (f(r) + \alpha b, \langle c, r \rangle + \alpha\mu)$$

where:

- f is a (linear) endomorphism of R;
- $b \in R$ is a vector;

– $c \in R'$ is a non-zero vector of the dual;

– μ is a real number;

– f, b, c, μ are such that $\hat{\phi} \in GL(\hat{R})$.

The function $\hat{\phi}$ induces a projective transformation $\phi \in GL(\mathbf{P})$. In the following ϕ will be assumed to belong to $GL(\mathbf{P}, V)$. Note that the condition $c \neq 0$ implies that $\phi \notin GL(V)$.

Since V is open, convex and regular, then $\langle c, r\rangle + \alpha\mu \neq 0$ for all $(r, \alpha) \in V \times \mathbf{R}_*$; hence ϕ can be expressed as follows, on V:

$$\phi : v \mapsto \frac{f(v) + b}{\langle c, v\rangle + \mu} \qquad (v \in V). \tag{1}$$

The cone V being arcwise connected, the denominator $v \mapsto \langle c, v\rangle + \mu$ (which is non-vanishing on V) has always the same sign on V. Hence we can assume $\langle c, v\rangle + \mu > 0$ for all $v \in V$.

The following can be proved:

– $c \in \overline{V}' \setminus \{0\}$ and $\mu \geq 0$;

– $f(V) \subset \overline{V}$;

– f cannot be an isomorphism of R;

– $f(V) \subset \partial V$;

– $f(V)$ is a convex, regular cone, open with respect to the induced topology of the hyperplane $\mathrm{Im}(f)$;

– V can be written as

$$V = f(V) \oplus R_*^+ b \tag{2}$$

(implying $\mu = 0$ in (1)).

Hence ϕ is expressed by:

$$\phi(v) = \frac{f(v) + b}{\langle c, v\rangle} \qquad (v \in V).$$

Let k be the smallest natural number such that f restricted to the subspace $\mathrm{Im}(f^k)$ is an isomorphism of $\mathrm{Im}(f^k)$. Similar arguments to the one leading to (2) yield:

$$V = f^k(V) \oplus \mathbf{R}_*^+ f^{k-1}(b) \oplus \cdots \oplus R_*^+ b \tag{3}$$

and

$$\phi(u + x_k f^{k-1}(b) + \cdots + x_1 b) = \frac{f(u) + x_{k-1} f^{k-1}(b) + \cdots + x_1 f(b) + b}{x_k \langle c, f^{k-1}(b)\rangle}.$$

(Note that in (3) it may be $f^k(V) \equiv f^k(R) \equiv \{0\}$).

The following facts provide an answer to the problem of the existence of projective automorphisms for V, supplying also a description of their structure.

THEOREM 2.1. *Let* V *be an open, convex, regular cone of the vector space* R. *The group of projective automorphisms of* V *strictly contains the group of all linear automorphisms of* V, *if, and only if, the maximal decomposition of the cone* V *contains at least one half-line.*

In the latter case, let

$$V = W \oplus R_*^+ b \qquad (b \in R)$$

(of course W may be the cone {0} of the 0–dimensional vector space); then

$$GL(\mathbf{P}, V) = GL(V) \cdot Q \cdot GL(V)$$

where $Q = \{I, p\}$ is the group consisting of:

$$I = \text{identity of } V$$

and

$$p \text{ defined by: } w + xb \mapsto \frac{w+b}{x} .$$

Let $(GL(V))_I$ and $(GL(\mathbf{P}, V))_I$ denote respectively the connected components of the identity of GL (V) and GL (**P**, V) in GL (**P**). Then

$$(GL(V))_I = (GL(\mathbf{P}, V))_I .$$

In other words, the (image of the) Lie algebra of GL (V) coincides with the Lie algebra of GL (**P**, V).

Let $\dim R > 1$. The above defined automorphism p is not an isometry of V (with respect to the canonical Riemannian metric) and it cannot be extended to a holomorphic automorphism of the tube domain T (V).

Hence:

THEOREM 2.2. *Let* $V \subset R$ *be an open, convex, regular cone, with* $\dim R > 1$, *and let* ϕ *be an element of* GL (**P**, V).

Then ϕ *is an isometry of* V *(with respect to the canonical Riemannian metric) if, and only if,* $\phi \in GL(V)$.

THEOREM 2.3. *Let* $V \subset R$ *be an open, convex, regular cone, with* $\dim R > 1$, *and let* ϕ *be an element of* GL (**P**, V).

Then ϕ *can be extended to a holomorphic automorphism of the tube domain associated to* V, *if, and only if,* $\phi \in GL(V)$.

If V is a half-line, then GL (**P**, V) coincides with the group of isometries (with respect to the canonical Riemannian metric) of V, and each of its elements can be extended to a holomorphic automorphism of the corresponding tube domain (the upper half-plane) (see [1]).

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