Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Marcus froim

On a result of S.S. Chern concerning contact invariants between a pair of surfaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **68** (1980), n.6, p. 488–491.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1980_8_68_6_488_0>

Geometria differenziale. — *On a result of S. S. Chern concerning contact invariants between a pair of surfaces.* Nota di FROIM MARCUS, presentata (*) dal Socio G. FICHERA.

RIASSUNTO. — Si dimostra usando il metodo di Fubini che le due invarianti di contatto fra due superficie che furono trovate da S. Chern, usando il metodo di Cartan, e che a lui parevano essere nuove, sono le invarianti di contatto in una corrispondenza asintotica ottenute da Čhech.

1. The notion of geometric and analytical contact between curves, surfaces or varieties was introduced by Fubini in [1], see also [2]. In [3], M. Mascalchi (1) presented a geometric interpretation of the contact invariants between a pair of surfaces S_1, S_2 having the same tangent plane at a common point. Let the pair of surfaces be

$$(1.1) \qquad z = a_1 x^2 + 2\, b_1 xy + c_1 y^2 + (3) \quad ; \quad z = a_2 x^2 + 2\, b_2 xy + c_2 y^2 + (3)$$

with a_i, b_i, c_i constants, and with the figure (n) in parantheses denoting quantities comprising terms in x, y of order n at least; the contact invariants are

$$(1.2) \qquad \frac{a_2 c_2 - b_2^2}{a_1 c_1 - b_1^2} \quad ; \quad \frac{a_2 c_1 - 2\, b_1 b_2 + a_1 c_2}{a_1 c_1 - b_1^2} ,$$

the first being the ratio of total curvatures of S_2 and S_1 at their common point (the origin), the second—the ratio of the harmonizant of:

$$(1.3) \qquad a_1 x^2 + 2\, b_1 xy + c_1 y^2 = 0 \quad ; \quad a_2 x^2 + 2\, b_2 xy + c_2 y^2 = 0 ,$$

and the total curvature of S_1 at zero.

Chern, using Cartan's method, obtained [4] (2) two further contact invariants—which he believed to be new—between a pair of surfaces with second-order contact a common point. Apart from that, as the bibliography in G. Bol's excellent treatise on projective differential geometry [5] indicates, there has been no further study of Chern's invariants. Accordingly, an attempt is made

(*) Nella seduta del 26 giugno 1980.

(1) M. Mascalchi was assistant to our master G. Fubini when I was a student at the Turin Polytechnic.

(2) I became aware of this paper through the selected papers of S. S. Chern.

in the present paper to establish their significance and their relationship to the theory of asymptotic correspondence between a pair of unruled surface, using Fubini's method [6].

2. In Chern's paper, the equations of the pair of surfaces with the same tangent plane at a common point A, read:

$$z = xy + \frac{1}{3}(px^3 + 3\,qx^2y + 3\,rxy^2 + sy^3) + (4)\,, \tag{2.1}$$

$$z = xy + \frac{1}{3}(p_1x^3 + 3\,q_1x^2y + 3\,r_1xy^2 + s_1y^3) + (4)\,. \tag{2.2}$$

If $AA_1A_2A_3$ is the reference frame in which (2.1) and (2.2) are the equations of S_1 and S_2, and AA_1A_2 their common tangent plane, then the most generalized reference frame $\bar{A}\bar{A}_1\bar{A}_2\bar{A}_3$ having the same property is given by:

$$\bar{A} = A\,,\ \bar{A}_1 = \rho A + \alpha A_1\,,\ \bar{A}_2 = \sigma A + \delta A_2\,,\ \bar{A}_3 = \tau A + \lambda A_1 + \mu A_2 + \alpha\delta A_3\,, \tag{2.3}$$

or

$$\bar{A} = A\,,\ \bar{A}_1 = \rho A + \alpha A_2\,,\ \bar{A}_2 = \sigma A + \delta A_1\,,\ \bar{A}_3 = \tau A + \mu A_1 + \lambda A_2 + \alpha\delta A_3\,, \tag{2.4}$$

where the operation (2.4) may be regarded as the product of (2.3) and of the transformation

$$\bar{A} = A\,,\ \bar{A}_1 = A_2\,,\ \bar{A}_2 = A_1\,,\ \bar{A}_3 = A_3\,. \tag{2.5}$$

If the equations of the surfaces (2.1) (2.2) in the reference frame $\bar{A}\bar{A}_1\bar{A}_2\bar{A}_3$ defined by (2.3), are:

$$z = xy + \frac{1}{3}(\bar{p}x^3 + 3\,\bar{q}x^2y + 3\,\bar{r}xy^2 + \bar{s}y^3) + (4)\,, \tag{2.6}$$

$$z = xy + \frac{1}{3}(\bar{p}_1x^3 + 3\,\bar{q}_1x^2y + 3\,\bar{r}_1xy^2 + \bar{s}_1y^3) + (4)\,, \tag{2.7}$$

it follows that:

$$\frac{\bar{p}_1}{\bar{p}} = \frac{p_1}{p}\ ;\quad \frac{\bar{s}_1}{\bar{s}} = \frac{s_1}{s}\,. \tag{2.8}$$

On the other hand, the operation (2.5) transforms p_1/p and s_1/s according to:

$$\frac{\bar{p}_1}{\bar{p}} = \frac{s_1}{s}\ ;\quad \frac{\bar{s}_1}{\bar{s}} = \frac{p_1}{p}\,. \tag{2.9}$$

Accordingly, Chern observed that the quantities

$$\text{(c)} \qquad H = \frac{1}{2}\left(\frac{p_1}{p} + \frac{s_1}{s}\right) \quad ; \quad K = \frac{p_1}{p} \cdot \frac{s_1}{s} ,$$

are independent of the choice of the reference frame $AA_1\, A_2\, A_3$, and are contact invariants of the surfaces (2.1), (2.2) at the common point A. In concluding his paper Chern, using the method of C. Segre [6], gave a geometric interpretation *not of* H *and* K *but of* p_1/p *and* s_1/s.

3. From (2.1) and (2.2), it follows that the asymptotes of the two surfaces correspond. We may suppose that their common point is o, and that $x = x(u, v)$, $y = y(u, v)$. Let $x = z = 0$ be the tangent to the asymptote $v = \text{const}$, and $y = x = 0$ the tangent to $u = \text{const}$. Then at o:

$$\text{(3.1)} \qquad x = y = z = 0 \quad ; \quad x_v = y_u = 0 .$$

From (2.1), it follows that for o we have:

$$\text{(3.2)} \qquad \begin{aligned} &z_x = z_y = z_{xx} = z_{yy} = 0 \quad ; \quad z_{xy} = 1 , \\ &z_{xxx} = 2p \quad ; \quad z_{xxy} = 2q \quad ; \quad z_{xyy} = 2r , z_{yyy} = 2s , \end{aligned}$$

whence, again for o,

$$\text{(3.3)} \qquad z_u = z_v = 0 .$$

Consider the fundamental system (see [6]):

$$\text{(3.4)} \qquad x_{uu} = \theta_u x_u + \beta x_v \quad ; \quad x_{vv} = \gamma x_u + \theta_v x_v .$$

Therefore, we have for o

$$\text{(3.5)} \qquad \begin{aligned} &x_{vv} = \gamma x_u \quad ; \quad y_{uu} = \beta y_v \quad ; \quad z_{uu} = z_{vv} = 0 \quad ; \quad z_{uv} = x_u y_v , \\ &z_{uuu} = \beta z_{uv} \quad ; \quad z_{vvv} = \gamma z_{uv} . \end{aligned}$$

But from (2.1) and (3.1) it follows that for the same point:

$$\text{(3.6)} \qquad z_{uuu} = 2\, p x_u^3 + 3\, \beta x_u y_v \quad ; \quad z_{vvv} = 2\, s y_v^3 + 3\, \gamma x_u y_v .$$

Equating (3.6) and (3.5), we obtain:

$$(\alpha) \qquad p x_u^3 = -\beta x_u y_v \quad ; \quad s y_v^3 = -\gamma u_u y_v .$$

(In the same manner, we obtain from (2.2) for S_2:

$$(\delta) \qquad p_1 x_u^3 = -\beta_1 x_u y_v \quad ; \quad s_1 y_v^3 = -\gamma_1 x_u y_v ,$$

whence

$$(\mathscr{D}) \qquad \frac{p_1}{p} = \frac{\beta_1}{\beta} \quad ; \quad \frac{s_1}{s} = \frac{\gamma_1}{\gamma} .$$

But according to Čech [6], if a pair of surfaces determined by their Fubini linear elements

$$(3.7) \qquad \frac{\beta \mathrm{d}u^3 + \gamma \mathrm{d}v^3}{2\,\mathrm{d}u\,\mathrm{d}v} , \qquad \text{and} \quad \frac{\beta_1\,\mathrm{d}u^3 + \gamma_1\,\mathrm{d}v^3}{2\,\mathrm{d}u\,\mathrm{d}v} ,$$

are in asymptotic correspondence, then

$$(\mathrm{E}) \qquad \frac{\beta_1}{\beta} \quad \text{and} \quad \frac{\gamma_1}{\gamma} ,$$

are contact invariants of their corresponding asymptotes, and since a projective transformation of a surface conserves β *and* γ, *the contact invariants of* (2.1) *and* (2.2) *are given by* $\frac{p_1}{p}$ *and* $\frac{s_1}{s}$. *Changing x to y we obtain* $\frac{p_1}{p} = \frac{\gamma_1}{\gamma}$ *and* $\frac{s_1}{s} = \frac{\beta_1}{\beta}$. *Now compare with* (2.9).

The results explain why p_1/p *and* s_1/s *are the limits of the anharmonic ratios found by Chern* (3). On the basis of these results, Chern gives a very interesting geometric interpretation of the ratios β_1/β, γ_1/γ over and above the interpretation given by Čech.

BIBLIOGRAPHY

[1] G. FUBINI (1916) – *Applicabilità proiettiva di due superfici*, *41*, 135–162, Palermo.

[2] E. BOMPIANI (1965–1966) – *Dopo cinquant'anni dall'inizio della géométria proiettiva differenziale secondo* G. Fubini, « Rendiconti del Seminario Matematico » Università e Politecnico di Torino, *25*, 83–106.

[3] M. MASCALCHI (1934) – *Un nuovo invariante proiettivo di contatto di due superficie*, « Boll. Unione Mat. Italiana », *13*, 45–59.

[4] S. SHEN CHERN (1941) – *Sur les invariànts de contact en géométrie projective différentielle*, « Pont. Acad. Sci. », Acta 5, 123–140.

[5] G. BOL (1954–1967) – *Projektive Differentialgeometrie*, « Studia Mathematica » *1*, *2*, *3* Vanderrhoeck and Ruprecht in Göttingen.

[6] G. FUBINI and E. ČECH (1931) – *Introduction a la géométrie projective differentielle des surfaces*, Gauthier-Villars, Paris.

(3) *Let there be a quadric* Σ *having second-order contact with a pair of surface at a common point* A, *and an arbitrary point* Q *of* Σ. *An arbitrary straight line* d, *passing through* Q, *meets* Σ *and the surfaces at points* P , M , N. *When* d *tends to* QA *on the asymptotic directions, the anharmonic ratio* (MN , QP , *tends to* p_1/p *and* s_1/s *for all choices of* Σ *and* Q.