Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Attilio Le Donne, Maria Vittoria Marchi

Representation of Lipschitzian compact-convex valued mappings

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **68** (1980), n.4, p. 278–280.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1980_8_68_4_278_0>

Analisi matematica. — *Representation of Lipschitzian compact-convex valued mappings.* Nota di ATTILIO LE DONNE e MARIA VITTORIA MARCHI, presentata (*) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Parametrizzazione di una mappa multivoca lipschitziana a valori compatti e convessi. La parametrizzazione è ottenuta mediante una famiglia di funzioni lipschitziane indicizzate in uno spazio compatto.

1. INTRODUCTION

Purpose of the present paper is to investigate the so called representation problem: assume that we are given a compact-convex valued map F from a topological space X into $\mathbf{R}^n$. We ask whether there exists a compact $\mathcal{U}$ and a continuous single-valued map $f : X \times \mathcal{U} \to \mathbf{R}^n$ such that

$$F(x) = f(x, \mathcal{U})$$

and f inherits the smoothness properties of F.

In the case where F is continuous, this has been proved by Ekeland and Valadier [2]. In what follows we assume that F is Lipschitzian from a metric space X and we prove the existence of $\mathcal{U}$ and f such that f is Lipschitzian in x and the above representation formula holds.

2. In the following we denote by X a metric space and by Ω^n the space of compact-convex subsets of $\mathbf{R}^n$ provided with the Hausdorff metric D induced by the Euclidean norm $|\cdot|$. For $x \in \mathbf{R}^n$ and $\varepsilon \geq 0$ we denote by $B(x, \varepsilon)$ the closed ε-ball centered in x.

Let $\alpha, \beta, L \in \mathbf{R}^+$ be the solutions of the system

$$\begin{cases} \alpha + \beta = \pi/3 \\ \sin \alpha = 1/L \\ \sin \beta = 2/L \\ \alpha < \beta < \pi/2 . \end{cases}$$

(*) Nella seduta del 12 aprile 1980.

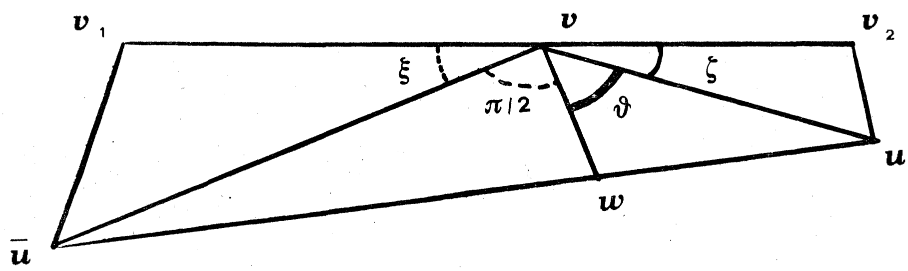
LEMMA 1. *Let* $\bar u, u, v_1, v_2, v \in \mathbf{R}^n$ *and* $\varepsilon, \eta \in \mathbf{R}^+$ *be such that*

$$\begin{aligned} &|\bar u - v_1| \leq \eta \\ &|u - v_2| \leq \varepsilon \\ &v \in [v_1, v_2] \quad \textit{and} \quad |\bar u - v| = 2\eta \\ &|u - v| > L\varepsilon \end{aligned}$$

then

$$|\bar u - u| > 2(\varepsilon + \eta).$$

Proof. Since the minimum of $|\bar u - u|$ is attained when $\bar u$ and u lie in the same half-plane with respect to the line through v_1 and v_2, there is no loss of generality in assuming that $\bar u$ and u satisfy the above condition of lying on the same half-plane.



For $\xi = \widehat{v_1 v \bar u}$ and $\zeta = \widehat{u v v_2}$ we have $\xi + \zeta \leq \pi/2$. Indeed $\xi \leq \pi/6$ since $\sin \xi \leq \eta/2\eta$ and $\zeta \leq \alpha$ since $\sin \zeta \leq \varepsilon/L\varepsilon$, so there exists a point $w \in [\bar u, u]$ such that $\widehat{\bar u v w} = \pi/2$. Thus for $\vartheta = \widehat{w v u}$ we get $\beta \leq \vartheta \leq \pi/2$ and so $\sin \beta \leq \sin \vartheta$. From this it follows that

$$\begin{aligned} |\bar u - u| = |\bar u - w| + |u - w| \geq 2\eta + |u - v| \sin \vartheta \geq \\ \geq 2\eta + |u - v| \sin \beta \geq 2\eta + L\varepsilon\, 2/L = 2(\varepsilon + \eta). \quad \square \end{aligned}$$

To prove our main theorem we use the following result due to Bressan [1].

LEMMA 2. *There exists a Lipschitzian map* $b : \Omega^n \to \mathbf{R}^n$, *with Lipschitz constant not larger than* $2n$, *such that* $b(K) \in K$ *for every* $K \in \Omega^n$. $\square$

THEOREM 3. *For* $\overline{K} \in \Omega^n$ *and* $\bar u \in \overline{K}$ *there exists a Lipschitzian map* S *from* Ω^n *into* $\mathbf{R}^n$ *such that* $S(K) \in K$ *for every* $K \in \Omega^n$ *and* $S(\overline{K}) = \bar u$, *with Lipschitz constant not larger than* $2nL$.

Proof. Define $S(K) = b(U(K))$ where b is as in Lemma 2 and $U : \Omega^n \to \Omega^n$ is defined by $U(K) = K \cap B(\bar u, 2D(\overline{K}, K))$.

To prove the Theorem we show that U is Lipschitzian with Lipschitz constant less than or equal to L. For $K, K' \in \Omega^n$ and $u \in U(K)$ there exist $v_1, v_2 \in K'$ such that

$$| \bar{u} - v_1 | \leq D(\bar{K}, K')$$

$$| u - v_2 | \leq D(K, K')$$

Since $L > 1$, when $v_2 \in B(\bar{u}, 2D(\bar{K}, K'))$ we have

$$| u - v_2 | \leq LD(K, K').$$

with $v_2 \in U(K')$.

On the other hand, when $v_2 \notin B(\bar{u}, 2D(\bar{K}, K'))$, then exists a point $v \in [v_1, v_2]$ such that $| \bar{u} - v | = 2D(\bar{K}, K')$.

For such a v, that belongs to $U(K')$, it must be that

$$| u - v | \leq LD(K, K')$$

since the inequality

$$| \bar{u} - u | \leq 2D(\bar{K}, K) \leq 2(D(\bar{K}, K') + D(K, K'))$$

contradicts the conclusion of Lemma 1.

Since K, K' and u were arbitrary, we conclude

$$D(U(K), U(K')) \leq LD(K, K'). \quad \square$$

COROLLARY 4. *Let* F *be a compact-convex valued map from a metric space* X *into* R^n, *Lipschitzian with Lipschitz constant* M. *Then there exist a compact space* $\mathcal{U}$ *and a continuous map* $f: X \times \mathcal{U} \to R^n$ *such that* f *is Lipschitzian in* x *and the following representation formula holds*:

$$F(x) = f(x, \mathcal{U}) \qquad \textit{for each} \quad x \in X.$$

Proof. We take as $\mathcal{U}$ the space of all Lipschitzian selections of F with Lipschitz constant not larger than $2nML$ provided with the pointwise topology. $\mathcal{U}$ is compact since it is a closed subspace of the compact space $\prod_{x \in X} F(x)$.

As $f: X \times \mathcal{U} \to \mathbf{R}^n$ we choose the map defined by $f(x, V) = V(x)$. f is Lipschitzian in x and, since the family $\mathcal{U}$ is equicontinuous, it is continuous [3. Ch. 7 th. 15].

For $x \in X$ and $u \in F(x)$, let S be as in Theorem 3 where $\bar{u} = u$ and $\bar{K} = F(x)$. The map $S \circ F$ belongs to $\mathcal{U}$, hence $F(x) = f(x, \mathcal{U})$.

REFERENCES

[1] A. BRESSAN - *Misure di curvatura e selezioni lipschitziane*, to appear.

[2] I. EKELAND, M. VALADIER (1971) - *Representation of set-valued mapping*, « J. Math. Anal. Appl. », *35-3*, 621-629.

[3] J. L. KELLEY - *General Topology*, Van Nostrand 1955, ed. 1968.