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Some considerations on the global properties of thermodynamic processes in continua

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Meccanica dei continui. — *Some considerations on the global properties of thermodynamic processes in continua* (*). Nota di RAFFAELE ESPOSITO e ANTONIO ROMANO, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — R. Fosdick e J. Serrin hanno mostrato di recente, a partire dalla diseguaglianza termica (1.7), che alcuni risultati termodinamici dedotti da C. Truesdell nell'ambito dei processi omogenei sono altresì validi per processi non omogenei. In questa nota si prova che alla stessa estensione si perviene, anche senza la diseguaglianza termica (1.7), per ogni sistema termodinamico che sia parte di un *continuo* più ampio.

INTRODUCTION

In a recent paper [1] R. Fosdick and J. Serrin showed that Rational Continuum Thermodynamics, when it accepts the Clausius-Duhem inequality as a primitive assertion of the irreversibility of continuous processes, in particular cases implies the classical statements of the second law of Thermodynamics due to Carnot, Clausius, Kelvin and Planck.

More precisely, R. Fosdick and J. Serrin consider a deformable, heat conducting body $\mathscr{B}$ immersed in a *unspecified* environment $\mathscr{B}^e$ having uniform but possibly not constant temperature $\tau(t) > 0$ and subject to mass and surface loading and heating. Moreover they adopt *heat transfer inequalities* which are a mathematical expression of the statement " heat... never passes out of a body except when it flows by conduction or radiation into a colder body " (Maxwell [2], p. 154). Afterward, by using the global balance of energy for the whole body $\mathscr{B}$ and a remarkable identity, they derive an estimate (1) for the efficiency in an arbitrary process of $\mathscr{B}$ regarded as possibly irreversible heat engine. This estimate obtained, they prove the aforsaid implications of the Clausius-Duhem inequality. In fact their analysis does not require the full strength of the Clausius-Duhem inequality. It suffices to lay down as the main axiom what Truesdell and Muncaster ([6], Chapter I, Section (V)) call the *heat-bath inequality*.

In the present paper we observe (section 1) that *when the body $\mathscr{B}$ is regarded as a portion of a larger continuous one*, jump conditions derived from the balance of momentum and energy and from the Clausius-Duhem inequality

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(1) This result had already been obtained in a less direct way by C. Truesdell in [3] for systems subject to homogeneous processes.

do not always imply the heat transfer inequality. This is true, for instance, if the heat vector $\boldsymbol{h}$ is continuous across the boundary $\partial\mathscr{B}$ of $\mathscr{B}$ or, equivalently, if on $\partial\mathscr{B}$ dissipation produced by external surface loading, vanishes. To prove these statements, we consider only circumstances in which the heat supply $r = 0$. On the other hand, again considering only circumstances in which $r = 0$, we show (section 2) that the estimate of efficiency derived in [1] holds even if the aforesaid heat transfer inequality is not assumed. In fact it is a direct consequence of the balance of energy, the Clausius-Duhem inequality and jump conditions. This derivation of the estimate of efficiency of a heat engine leads to the conclusion that all the general considerations developed by C. Truesdell in [4] (2) for homogeneous processes are also valid for inhomogeneous ones, provided that in the formulas of [4] we replace the uniform temperature θ of the body by the uniform temperature τ of the environment.

1. Considerations on heat transfer inequality

Let $\mathscr{S}$ be a continuous body. The relations expressing balance of momentum, balance of energy and the Clausius-Duhem inequality in the absence of heat supply are as follows:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_t} \rho\dot{\boldsymbol{x}}\,\mathrm{d}\mathscr{C} = \int_{\partial\mathscr{P}_t} \mathbf{T}\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \int_{\mathscr{P}_t} \rho\boldsymbol{b}\,\mathrm{d}\mathscr{C}\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_t} \rho\left(\tfrac{1}{2}\dot{\boldsymbol{x}}^2 + \varepsilon\right)\mathrm{d}\mathscr{C} = \int_{\partial\mathscr{P}_t} (\dot{\boldsymbol{x}}\cdot\mathbf{T} - \boldsymbol{h})\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \int_{\mathscr{P}_t} \rho\dot{\boldsymbol{x}}\cdot\boldsymbol{b}\,\mathrm{d}\mathscr{C}\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_t} \rho\eta\,\mathrm{d}\mathscr{C} \geq -\int_{\partial\mathscr{P}_t} \frac{\boldsymbol{h}\cdot\boldsymbol{n}}{\theta}\,\mathrm{d}\sigma\,,
\end{aligned}
\tag{1.1}
$$

where $\mathscr{P}_t$ denotes any part of the body $\mathscr{S}$ in the present configuration $\mathscr{C}_t$, $\partial\mathscr{P}_t$ is the boundary of $\mathscr{P}_t$ whose exterior unit normal vector is $\boldsymbol{n}$, $\boldsymbol{x}$ is the present position of a particle $X \in \mathscr{S}$ and moreover:

ρ = mass density,
$\mathbf{T}$ = stress tensor,
$\boldsymbol{b}$ = specific body force,
ε = specific internal energy,
$\boldsymbol{h}$ = heat flux vector,
η = specific entropy,
θ = local temperature.

(2) See [5] too.

As is well known, the relations (1.1) imply the local equations of balance and the local Clausius-Duhem inequality at those points of $\mathscr{P}_t$ where the integrands are regular, while at the points of a surface $\sigma \subset \mathscr{P}_t$ which is singular for these functions, (1.1) imply the following jump conditions (see [7], (193.3)):

$$
\begin{aligned}
&[\rho U \dot{\boldsymbol{x}}] + [\mathbf{T}]\cdot\boldsymbol{n} = 0\,,\\
(1.2)\qquad &[\rho U(\tfrac{1}{2}\dot{\boldsymbol{x}}^2 + \varepsilon)] + [\dot{\boldsymbol{x}}\cdot\mathbf{T} - \boldsymbol{h}]\cdot\boldsymbol{n} = 0\,,\\
&[\rho U \eta] - \left[\frac{\boldsymbol{h}}{\theta}\right]\cdot\boldsymbol{n} \le 0\,.
\end{aligned}
$$

The notations used are as follows: if A^e and A^i are the limits of the quantity A at a point of $\partial\mathscr{B}$, taken on paths within $\mathscr{B}^e$ and $\mathscr{B}^i$ respectively then $[A] = A^e - A^i$; if u_n denotes the normal velocity of σ, then

$$
U = u_n - \dot{\boldsymbol{x}}\cdot\boldsymbol{n} \tag{1.3}
$$

is the local speed of propagation of σ.

In this paper we regard a body $\mathscr{B}$ as a proper part of a continuous body $\mathscr{S}$. Then the environment B^e of $\mathscr{B}$ is given by $\mathscr{B}^e = \mathscr{S} - \mathscr{B}$. Moreover, we suppose that $\mathscr{B}$ may slip on $\mathscr{B}^e$ without producing cavitation. In this way, $\partial\mathscr{B}^e$ is a singular surface on which $[\dot{\boldsymbol{x}}]\cdot\boldsymbol{n} = 0$ and $U^i = 0$. On $\partial\mathscr{B}^e$ the Stokes-Christoffel condition is satisfied (3):

$$
[\rho\, U] = 0
$$

and (1.2) can be written:

$$
\begin{aligned}
&[\mathbf{T}]\cdot\boldsymbol{n} = 0\,,\\
(1.4)\qquad &[\dot{\boldsymbol{x}}\cdot\mathbf{T} - \boldsymbol{h}]\cdot\boldsymbol{n} = 0\,,\\
&\left[\frac{\boldsymbol{h}}{\theta}\right]\cdot\boldsymbol{n} \ge 0\,.
\end{aligned}
$$

It is trivial to verify that $(1.4)_{1,2}$ imply the relation

$$
[\dot{\boldsymbol{x}}]\cdot\boldsymbol{t} - [\boldsymbol{h}]\cdot\boldsymbol{n} = 0 \tag{1.5}
$$

where $\boldsymbol{t} = \mathbf{T}\cdot\boldsymbol{n}$ denotes the surface force density that $\mathscr{B}^e$ exerts on $\mathscr{B}$. On the other hand, $(1.4)_3$ by (1.5) becomes

$$
[\dot{\boldsymbol{x}}]\cdot\boldsymbol{t} + \frac{\boldsymbol{h}^i\cdot\boldsymbol{n}}{\theta^i}(\theta^i - \theta^e) \ge 0\,. \tag{1.6}
$$

Eqs. (1.5) and (1.6) lead to the conclusion that: *in all the processes of* $\mathscr{B}\cup\mathscr{B}^e$ *in which* $[\boldsymbol{h}]\cdot\boldsymbol{n} = 0$ *(or equivalently* $[\dot{\boldsymbol{x}}]\cdot\boldsymbol{t} = 0$*) on* $\partial\mathscr{B}$*, the heat transfer*

(3) The condition $[\rho U] = 0$, together with $U^e = U^i = 0$ does not imply $[\rho] = 0$.

inequality

$$\boldsymbol{h}\cdot\boldsymbol{n}\,(\theta^{i}-\theta^{e})\geq 0 \tag{1.7}$$

is satisfied as a consequence of the jump conditions (1.2). In other words, in these processes heat flows into colder regions. In particular, the condition $[\boldsymbol{h}]\cdot\boldsymbol{n}=0$ on $\partial\mathscr{B}$ holds for every process of $\mathscr{B}\cup\mathscr{B}^e$ if $\mathscr{B}^e$ is a perfect fluid. Indeed, on this hypothesis $\boldsymbol{t}=-p\boldsymbol{n}$ so that the assertion follows from the condition $[\dot{\boldsymbol{x}}]\cdot\boldsymbol{n}=0$ and (1.5). Inequality (1.7) is also satisfied when $\mathscr{B}$ does not slip on $\mathscr{B}^e$.

When the mechanical interaction between $\mathscr{B}$ and $\mathscr{B}^e$ includes friction on $\partial\mathscr{B}$, then (the principles of balance alone do not imply that (1.7) holds, but of course do not exclude it.

2. Considerations on the global thermodynamic relations

In this section we derive the classical statements of irreversibility from the Clausius-Duhem inequality without assuming the heat transfer inequality. We shall always suppose that $r=0$ and that the system $\mathscr{B}$ immersed in environment $\mathscr{B}^e$ at uniform but not necessarily constant temperature which from now on will be denoted by τ instead of θ^e.

As a beginning, we write $(1.1)_{2,3}$ in a suitable form for the whole system $\mathscr{B}$. In $(1.1)_{2,3}$ let $\mathscr{P}_t\subset\mathscr{B}_t$ tend to $\mathscr{B}_t$. Then if we take into account jump conditions (1.4), we derive the following global statement of the balance of energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{B}_t}\rho\left(\tfrac{1}{2}\dot{\boldsymbol{x}}^2+\varepsilon\right)\mathrm{d}\mathscr{C}=\int_{\partial\mathscr{B}_t}(\dot{\boldsymbol{x}}^e\cdot\boldsymbol{t}-\boldsymbol{h}^e\cdot\boldsymbol{n})\,\mathrm{d}\sigma+\int_{\mathscr{B}_t}\rho\dot{\boldsymbol{x}}\cdot\boldsymbol{b}\,\mathrm{d}\mathscr{C}\,, \tag{2.1}$$

and the heat bath inequality [4]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{B}_t}\rho\eta\,\mathrm{d}\mathscr{C}\geq-\frac{1}{\tau}\int_{\partial\mathscr{B}_t}\boldsymbol{h}^e\cdot\boldsymbol{n}\,\mathrm{d}\sigma\,. \tag{2.1$'$}$$

When we introduce the notations

$$\mathrm{E}(\mathscr{B}_t)\equiv\int_{\mathscr{B}_t}\rho\varepsilon\,\mathrm{d}\mathscr{C}\quad,\quad\mathrm{H}(\mathscr{B}_t)\equiv\int_{\mathscr{B}_t}\rho\eta\,\mathrm{d}\mathscr{C}\,,$$

$$\mathrm{W}(\mathscr{B}_t,\mathscr{B}_t^e)\equiv\int_{\partial\mathscr{B}_t}\dot{\boldsymbol{x}}^e\cdot\boldsymbol{t}\,\mathrm{d}\sigma+\int_{\mathscr{B}_t}\rho\dot{\boldsymbol{x}}\cdot\boldsymbol{b}\,\mathrm{d}\mathscr{C}-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{B}_t}\rho\,\tfrac{1}{2}\,\dot{\boldsymbol{x}}^2\,\mathrm{d}\mathscr{C}\,, \tag{2.2}$$

$$\mathrm{Q}(\mathscr{B}_t,\mathscr{B}_t^e)\equiv-\int_{\partial\mathscr{B}_t}\boldsymbol{h}^e\cdot\boldsymbol{n}\,\mathrm{d}\sigma\,,$$

(4) See [6], p. 15.

(2.1) become

$$\dot{\mathrm{E}}(\mathscr{B}_t) = \mathrm{W}(\mathscr{B}_t, \mathscr{B}_t^e) + \mathrm{Q}(\mathscr{B}_t, \mathscr{B}_t^e), \tag{2.3}$$
$$\dot{\mathrm{H}}(\mathscr{B}_t) \geq \frac{\mathrm{Q}(\mathscr{B}_t, \mathscr{B}_t^e)}{\tau}.$$

Relation $(2.3)_2$ is interesting because it coincides with the corresponding one derived by R. Fosdick and J. Serrin in [1]. Moreover, here it is deduced without assuming (1.7), but as a direct consequence of the jump conditions (1.4). When $\mathscr{B}^e$ does not slip along $\mathscr{B}$, we have $[\dot{\boldsymbol{x}}]\cdot\boldsymbol{t} = 0$ (or equivalently $[\boldsymbol{h}]\cdot\boldsymbol{n} = 0$) and the functions $\dot{\boldsymbol{x}}\cdot\boldsymbol{t}$ and $\boldsymbol{h}\cdot\boldsymbol{n}$ are continuous across $\partial\mathscr{B}$. In this case internal determinations of $\dot{\boldsymbol{x}}$ and $\boldsymbol{h}$ can be used in applying the definition $(2.2)_{2,3}$. Relations (2.3) permit to find again the estimate of efficiency deduced in [1] and [4].

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REFERENCES

[1] R. FOSDICK and J. SERRIN (1975) – *Global properties of continuum thermodynamic processes*, « Arch. Rat. Mech. Anal. », *59*, 97–109.

[2] J. C. MAXWELL (1872) – *Theory of heat*, Second edition, London, Longmans Press & Co.

[3] C. TRUESDELL (1976) – *Termodinamica Razionale*, « Contributi del Centro Linceo interdisciplinare », n. 20.

[4] G. TRUESDELL (1976) – *Improved estimates of efficiency of irreversible heat engines*, « Ann. di Matematica », *108*, 305–323.

[5] C. TRUESDELL (1977) – *Macchine termiche irreversibili e la seconda legge della Termodinamica* « Problemi attuali di Meccanica teorica e applicata », Convegno di Torino in accordo con Accademia delle Scienze – 297–307.

[6] C. TRUESDELL and R. G. MUNCASTER (1979) – *Fundamentals of Maxwell's kinetic theory of a simple monatomic gas, treated as a branch of rational mechanics* », « Academic Press », New York.

[7] C. TRUESDELL and R. TOUPIN (1960) – *The Classical field theories*, « Handbuch der Physik », Band III/1, Springer Verlag, Berlin, New-York.