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Differential Geometry of Light-Cones

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Geometria. — *Differential Geometry of Light-Cones.* Nota di
GRAZIANO GENTILI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studia la geometria Riemanniana invariante dei coni-luce di $\mathbf{R}^n$. Si determinano tutte le isometrie di un tale cono e si discute la loro estendibilità olomorfa al dominio tubolare associato al cono.

A cone in $\mathbf{R}^n$ is called *regular* if it contains no affine line. Let $V \subset \mathbf{R}^n$ be an open convex regular cone, and let V′ be the dual cone of V, i.e. the cone of linear forms x' on $\mathbf{R}^n$ such that $\langle x, x' \rangle > 0$ for all $x \in \overline{V} - \{0\}$; V′ is also an open convex regular cone of $\mathbf{R}^n$. Henceforth, all cones will always be assumed to be open convex and regular.

The *characteristic* function (see [10]) of the cone V is the C^∞ function (defined on V)

$$\Phi_V(x) = \int_{V'} \exp(-\langle x, x' \rangle)\, dx' \qquad (x \in V)$$

where dx' is the Lebesgue measure on $\mathbf{R}^n$. The function $\log \Phi_V$ is strictly convex, hence the quadratic differential form

$$\sum_{i,j=1}^{n} \frac{\partial^2 \log \Phi_V}{\partial x_i\, \partial x_j}(x)\, dx_i\, dx_j$$

defines a (positive definite) Riemannian metric of class C^∞ on V. Since every linear automorphism Ω of the cone V is such that

$$|\det \Omega| \cdot (\Phi_V(\Omega x)) = \Phi_V(x) \qquad (x \in V) \tag{1}$$

then the above defined Riemannian metric is invariant under the action of the group GL (V) of all affine (hence linear, see [10]) automorphisms of V. In the following work, we will consider only this invariant Riemannian metric of V.

The linear form $-d(\log \Phi_V)_x$ is a point in V′, which will be denoted by $\star x$. The function $x \mapsto \star x$ defines a C^∞ diffeomorphism of V onto V′. Moreover, if V is affine-homogeneous (i.e., GL (V) acts transitively on V), then $\star(\star x) = x$ for all $x \in V$, and also the diffeomorphism $x \mapsto \star x$ is an isometry of V onto V′ (with respect to the invariant metrics of V and V′). References for these facts are [7], [9], [10].

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The following problems arise naturally.

The first concerns the determination of the whole group of isometries for the invariant Riemannian metric of a cone.

Let $T(V) = \{x + iy : x \in \mathbf{R}^n, y \in V\} \subset \mathbf{C}^n$ be the tube domain associated to the cone V; the cone V will be identified with the subset $iV = \{iy : y \in V\}$ of $T(V)$. The second question deals with the extensibility of the isometries (for the Riemannian metric of V) to holomorphic automorphisms of the tube domain $T(V)$. In other words the question is: which isometries of V can be obtained as restriction to iV of biholomophic automorphisms of $T(V)$ leaving iV invariant?

This article will report some results obtained for the class of light-cones. This is an ample class of affine-homogeneous cones in $\mathbf{R}^n$, which are self-adjoint, in the sense that the canonical inner product defining the Euclidean distance in $\mathbf{R}^n$ identifies these cones with their duals. In dimension $n \geq 2$, the light-cone of $\mathbf{R}^n$ consists of all $(x_1, \cdots, x_n) \in \mathbf{R}^n$ such that $x_1^2 - \cdots - x_n^2 > 0$ and $x_1 > 0$. We shall consider $\mathbf{R}_\star^+$ (the set of strictly positive real numbers) as the one-dimensional light-cone.

In n. 1 the light-cones of dimension $n \geq 3$ will be realized as cones of " matrices " L_n^+. In terms of this realization we will compute the characteristic function, the Riemannian distance, the mapping $\star$, and we will describe the geodesic curves. Section 2 deals with the construction of the entire group of isometries for the light-cones of dimension $n \geq 3$. Section 3 solves the extensibility problem. Finally section 4 deals with the one—and two—dimensional light-cones for which a different approach has to be divised.

Proofs and further details will appear elsewhere.

1. *The light-cone* L_n^+ $(n \geq 3)$ *and its invariant Riemannian metric.*

Let L_n $(n \geq 3)$ be the set whose elements are the matrices $\begin{pmatrix} x & \boldsymbol{z} \\ \boldsymbol{z} & y \end{pmatrix}$ with $x, y \in \mathbf{R}$ and $\boldsymbol{z} \in \mathbf{R}^{n-2}$. This set, with addition and multiplication defined in the obvious way, is a real vector space of dimension $n \geq 3$. Let $h = \begin{pmatrix} x & \boldsymbol{z} \\ \boldsymbol{z} & y \end{pmatrix}$ and $k = \begin{pmatrix} t & \boldsymbol{w} \\ \boldsymbol{w} & u \end{pmatrix}$ be elements of L_n, and let us define

$$h \cdot k = \begin{pmatrix} xt + \boldsymbol{z} \cdot \boldsymbol{w} & x\boldsymbol{w} + u\boldsymbol{z} \\ t\boldsymbol{z} + y\boldsymbol{w} & uy + \boldsymbol{z} \cdot \boldsymbol{w} \end{pmatrix}$$

$$h \circ k = \tfrac{1}{2}(h \cdot k + k \cdot h) \in L_n .$$

$(L_n, \circ)$ is a (non associative) commutative algebra with identity $I = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$; in particular it turns out to be a Jordan algebra (see [4], [6]).

The bilinear map

$$(2) \qquad \begin{aligned} &(\, , \,) : \mathrm{L}_n \times \mathrm{L}_n \to \mathbf{R} \\ &(h \, , k) \mapsto tr\,(h \cdot k) = xt + uy + 2\,\boldsymbol{z} \cdot \boldsymbol{w} \end{aligned}$$

turns out to be a scalar product and, if $\| \quad \|$ denotes the norm it induces on L_n, the following relations hold, for a suitable $r \in \mathbf{R}_\star^+$ (see [6])

$$(3) \qquad \begin{aligned} &\| h \circ k \| \leq r \| h \| \cdot \| k \| \\ &\| h^m \| \leq r^m \| h \|^m \qquad\qquad \forall h \, , k \in \mathrm{L}_n \, , \quad \forall m \in \mathbf{N} \, . \end{aligned}$$

Formulas (3) allow us to define both the C^∞ function exp, from $(\mathrm{L}_n \, , \circ)$ in itself, $h \overset{\exp}{\mapsto} \sum_{m=0}^{\infty} \frac{h^m}{m!}$, and its inverse function log, defined on $\exp\,(\mathrm{L}_n)$.

If $q \in \mathrm{O}\,(n-2)$ is an orthogonal matrix, then the linear map

$$(4) \qquad \begin{aligned} &\mathrm{T}_q : \mathrm{L}_n \to \mathrm{L}_n \\ &\begin{pmatrix} x & \boldsymbol{z} \\ \boldsymbol{z} & y \end{pmatrix} \mapsto \begin{pmatrix} x & q \cdot \boldsymbol{z} \\ q \cdot \boldsymbol{z} & y \end{pmatrix} \end{aligned}$$

is an algebra-automorphism of $(\mathrm{L}_n \, , \circ)$.

If $a \in \mathrm{GL}\,(2 \, , \mathbf{R})$, $\boldsymbol{z} = (z_3 \, , \cdots , z_n) \in \mathbf{R}^{n-2}$, $\boldsymbol{w} = (w_3 \, , \cdots , w_n) \in \mathbf{R}^{n-2}$, let us define

$$(5) \qquad \begin{aligned} &g_a : \mathrm{L}_n \to \mathrm{L}_n \\ &\begin{pmatrix} x & \boldsymbol{z} \\ \boldsymbol{z} & y \end{pmatrix} \mapsto \begin{pmatrix} t & \boldsymbol{w} \\ \boldsymbol{w} & u \end{pmatrix} \end{aligned}$$

where $\begin{pmatrix} t & w_3 \\ w_3 & u \end{pmatrix} = a \begin{pmatrix} x & z_3 \\ z_3 & y \end{pmatrix} {}^t a \, ; \quad w_4 = z_4\,(\det a) \, ; \ldots \quad w_n = z_n\,(\det a)$;

the linear map g_a is an automorphism of the vector space L_n.

If $l_n \subset \mathrm{L}_n$ is the set of matrices $\begin{pmatrix} x & \boldsymbol{z} \\ \boldsymbol{z} & y \end{pmatrix}$ such that $\boldsymbol{z}$ has the last $n-3$ components equal to zero, then for all $h \in \mathrm{L}_n$, there exists $q \in \mathrm{O}\,(n-2)$ such that $\mathrm{T}_q\,(h) \in l_n$. In particular $\mathrm{T}_q\,(\exp\,(h)) = \exp\,(\mathrm{T}_q\,(h))$ and, if $\mathrm{T}_q\,(h) \in l_n$, $g_a \cdot \mathrm{T}_q\,(\exp\,(h)) = \exp\,(g_a \cdot \mathrm{T}_q\,(h))$ $(a \in \mathrm{O}\,(2))$. Analogous identities hold for the logarithm.

The " determinant " of the matrix $h = \begin{pmatrix} x & \boldsymbol{z} \\ \boldsymbol{z} & y \end{pmatrix}$ will be the real number

$xy - |z|^2$, denoted by $\det(h)$. Every $h = \begin{pmatrix} x & z \\ z & y \end{pmatrix} \in (L_n, \circ)$ such that $\det(h) \neq 0$ and that $xy > 0$, has a unique inverse element $h^{-1} \in (L_n, \circ)$, expressed by

$$h^{-1} = \frac{1}{xy - |z|^2} \cdot \begin{pmatrix} y & -z \\ -z & x \end{pmatrix}. \tag{6}$$

If we set, for $n \geq 3$,

$$L_n^+ = \left\{ h = \begin{pmatrix} x & z \\ z & y \end{pmatrix} \in L_n : \det(h) > 0, \quad x > 0 \right\}$$

then L_n^+ is a self adjoint (with respect to the scalar product defined in (2)) affine-homogeneous irreducible (see [9]) cone, isomorphic to the n-dimensional light-cone.

For all $q \in O(n-2)$ and for all $a \in GL(2, \mathbf{R})$ the functions T_q and g_a defined in (4), (5) are elements of $GL(L_n^+)$; besides that, for every $h \in L_n^+$, there exist $q \in O(n-2)$ and $a \in GL(2, \mathbf{R})$ such that $g_a T_q(h) = I$. Now, using essentially property (1), we obtain for the characteristic function of the cone L_n^+ (up to a positive constant factor)

$$\Phi_{L_n^+}(h) = (\det(h))^{-(n/2)} \qquad (h \in L_n^+).$$

The differential of $\log \Phi_{L_n^+}$ at the point $h \in L_n^+$ is represented by the vector $-\frac{n}{2} h^{-1}$ (see (2), (6)), hence

$$\star h = \frac{n}{2} h^{-1} \qquad (h \in L_n^+). \tag{7}$$

The unique fixed point of the involution $\star$ is $\sqrt{\frac{n}{2}}\, I$.

The problem of finding a geodesic curve joining any two points of a cone (see [9]) can be solved directly in this case, restricting the system of differential equations for geodesics to the subset D of diagonal matrices and integrating it. We obtain that, given any two points of L_n^+, there exists one and only one geodesic arc joining the two points (up to parametrization) (see [3]). Moreover this unique geodesic is a planar curve. The geodesic l joining I and the point $X \in L_n^+$ is given by

$$l(s) = \exp(s \cdot \log X) \qquad -\infty < s < \infty.$$

Integration of the "length element" along a geodesic arc lying in $D \cap L_n^+$, proves that the Riemannian distance from I, $d(I, \cdot)$, is

$$d(I, X) = \sqrt{\frac{n}{2}}\, \| \log X \| \qquad (X \in L_n^+).$$

2. *The group of isometries.*

In the case of light-cones of dimension $n \geq 3$, we determine the whole group of isometries. This is made possible by the explicit computation of the sectional curvature at a point of the cone.

$T_p(L_n^+)$ denotes the tangent space of L_n^+ at the point $p = \sqrt{\frac{n}{2}}\, I$. $T_p(L_n^+)$ is, of course, isomorphic to L_n and $\mathbf{R}^n$:

$$L_n \cong T_p(L_n^+) \ni \begin{pmatrix} \lambda_1 & (\lambda_3, \cdots, \lambda_n) \\ (\lambda_3, \cdots, \lambda_n) & \lambda_2 \end{pmatrix} = (\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n ,$$

Let Π be the two-dimensional section of $T_p(L_n^+)$ determined by the two vectors $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $\mu = (\mu_1, \cdots, \mu_n)$ of $T_p(L_n^+)$, and let

$$K(\Pi) = \frac{- R_{ijkl}\, \lambda_i\, \lambda_k\, \mu_j\, \mu_l}{(g_{ik}\, g_{jl} - g_{il}\, g_{jk})\, \mu_i\, \mu_k\, \lambda_j\, \lambda_l}$$

be the sectional curvature of Π (see [1]). It turns out that the only components of the Riemann tensor R_{ijkl}, with $i, k \leq 2$, $j, l \leq 3$, which do not vanish at p are $R_{1313} = R_{2323} = s = - R_{1323} = - R_{2313}$, where s is a certain positive constant depending only on n. The invariance of the metric yields

$$K(\Pi) = - s\,(1 - |P_{\Pi}(e)|^2) \tag{8}$$

where: e is a unit vector along the line $S = \{xI : x \in \mathbf{R}\} \subset L_n$; $P_{\Pi}(e)$ is the projection, with respect to the invariant Riemannian metric, of e on Π. In particular $K(\Pi) = 0$ if, and only if, Π contains S.

Formula (8) leads to the construction of the entire group of isometries for the Riemannian metric of the light-cone of $\mathbf{R}^n$, with $n \geq 3$. In fact, it can be proved that, if f is an isometry of L_n^+ keeping the points of S fixed, then there exist $a \in O(2)$ and $v, q \in O(n-2)$, such that $f = T_v\, g_a\, T_q$. Then (8) implies that, given any isometry g belonging to the isotropy subgroup of the point $p = \sqrt{\frac{n}{2}}\, I$, either g or $\star g$ leaves S pointwise invariant. Hence:

THEOREM 1. *The group of isometries of the light-cone L_n^+ $(n \geq 3)$ is the group*

$$GL(L_n^+) \cdot K \qquad (\cdot \text{ direct product})$$

where K consists of the identity and the involution $\star$. Moreover the connected component of the identity, for the group of isometries, is $GL(L_n^+)$.

3. *Extensibility.*

Every linear isometry of L_n^+ can be extended to $T(L_n^+)$ (see [5], [8]), and also the involution $\star$ is extensible (see [9]). It can be directly proved, however, that the holomorphic automorphism of $T(L_n^+)$

$$H \mapsto -\frac{n}{2} H^{-1} \qquad (H \in T(L_n^+))$$

is the extension of the involution (7). Hence:

THEOREM 2. *All the isometries of the cone L_n^+ are extensible as holomorphic automorphisms to the associated tube domain.*

4. $\mathbf{R}_\star^+$ *and* $\mathbf{R}_\star^+ \times \mathbf{R}_\star^+$.

In conclusion we shall discuss briefly the case of light-cones in dimension one and two.

In the case of $\mathbf{R}_\star^+$, the characteristic function, the metric tensor, and the involution are given by

$$\Phi_{\mathbf{R}_\star^+}(x) = \frac{1}{x} \quad ; \quad g(x) = \frac{1}{x^2} \quad ; \quad {}^\star x = \frac{1}{x} \qquad (x \in \mathbf{R}_\star^+).$$

The differential equation for an isometry is

$$\frac{f'(x)}{f(x)} = \pm \frac{1}{x} \qquad (x \in \mathbf{R}_\star^+)$$

yielding

$$f(x) = H \cdot x, \quad \text{or} \quad f(x) = \frac{K}{x} \qquad (H, K \in \mathbf{R}_\star^+).$$

Hence, the assertion of Theorem 1 (and Theorem 2) is valid also in this case.

The only light-cone in dimension two is the product $\mathbf{R}_\star^+ \times \mathbf{R}_\star^+$, and this case turns out to be exceptional. The characteristic function, the metric tensor, and the involution are

$$\Phi_{\mathbf{R}_\star^+ \times \mathbf{R}_\star^+}(x, y) = \frac{1}{xy}$$

$$(g_{ij}(x, y)) = \begin{pmatrix} \frac{1}{x^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \qquad (x, y) \in \mathbf{R}_\star^+ \times \mathbf{R}_\star^+.$$

$$\star(x, y) = \left(\frac{1}{x}, \frac{1}{y}\right).$$

Integration of the Killing equations (see [2])

$$\begin{cases} \dfrac{\partial \xi^1}{\partial x} - \dfrac{1}{x}\,\xi^1 = 0 \\[2ex] \dfrac{\partial \xi^2}{\partial y} - \dfrac{1}{y}\,\xi^2 = 0 \\[2ex] \dfrac{1}{x^2}\,\dfrac{\partial \xi^1}{\partial y} + \dfrac{1}{y^2}\,\dfrac{\partial \xi^2}{\partial x} = 0 \end{cases}$$

yields the splitting of the Lie algebra $\mathscr{G}$ of the group of isometries, as a direct sum of the 2-dimensional vector space $\mathscr{U}$ spanned by the vector fields

$$\alpha x \frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial y} \qquad ((\alpha , \beta) \in \mathbf{R}^2)$$

and of the 1-dimensional vector space $\mathscr{V}$ spanned by

$$(kx \cdot \log y)\,\frac{\partial}{\partial x} + (-\,ky \cdot \log x)\,\frac{\partial}{\partial y} \qquad (k \in \mathbf{R})\,.$$

Integration of the vector field of $\mathscr{V}$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = kx \cdot \log y \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\,ky \cdot \log x \end{cases} \qquad (k \in \mathbf{R})$$

gives the one-parameter subgroup of isometries

$$\Phi\,(t) : \mathbf{R}^+_\star \times \mathbf{R}^+_\star \rightarrow \mathbf{R}^+_\star \times \mathbf{R}^+_\star \qquad (t \in \mathbf{R})$$

$$(x\,,\,y) \mapsto (x^{\cos kt} \cdot y^{\sin kt}\,,\, y^{\cos kt} \cdot x^{-\sin kt})\,.$$

The orbit of a point q under the action of this subgroup is the (Riemannian) sphere with center (1,1) containing the point q. If $t = \pi/k$ we get

$$\Phi\,(\pi/k)\,((x\,,\,y)) = \left(\frac{1}{x}\,,\,\frac{1}{y}\right) = {}^\star(x\,,y)\,.$$

This shows easily that Theorem 1 is not valid in this case. Moreover the isometry $\Phi\,(t)$ is extensible to the tube domain if, and only if, $kt = \dfrac{\pi}{2}\,m$ $(m \in \mathbf{Z})$. Hence also the assertion of Theorem 2 cannot be generalized to this case.

Proof and further details will appear elsewhere.

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