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Tangent flag bundles and Jacobian varieties

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Geometria. — *Tangent flag bundles and Jacobian varieties.* Nota I di SAMUEL A. ILORI e AUBREY W. INGLETON, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Definiamo le sottovarietà «di Ehresmann» di un fascio di bandiere tangenti $V^{\triangle}$ sopra una varietà proiettiva algebrica irriducibile non – singolare, definita sopra un campo algebricamente chiuso. Poi mostriamo, usando una formula di intersezione, che le classi di cicli di tali sottovarietà «di Ehresmann» nell'anello di Chow di $V^{\triangle}$ possono essere determinate usando una conoscenza del più facile calcolo corrispondente in una varietà di bandiere $F(n+1)$. Questa teoria è poi applicata al calcolo delle classi di cicli di sottovarietà Jacobiane di V che sono definite mediante una famiglia indiciata di «nests» di sistemi lineari di «primals» in V.

1. INTRODUCTION

Let V be a non-singular irreducible algebraic projective variety (i.e. a subvariety of some projective space $\mathbf{P}_n$, say) of dimension d defined over an algebraically closed field k. The flag construction can be applied to the tangent bundle T of V to obtain an algebraic fibre bundle

$$\rho : V^{\triangle} \to V .$$

This bundle has as a fibre over $v \in V$, $F(d) = \{F = (F_0, F_1, \cdots, F_d) \mid \{v\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_d = T_v$, where F_i is a k-subspace and $\dim F_i = i\}$. $V^{\triangle}$ is called the tangent flag bundle of V and F, a tangent flag to V. The Chow ring of $V^{\triangle}$ is given by

$$A(V^{\triangle}) = \rho^* A(V)[\delta_1, \cdots, \delta_d]$$

subject to the relation

$$\prod_{h=1}^{d} (1 - \delta_h) = \rho^* c(V) .$$

(Cfr. Theorem 1, p. 4–19 in [3] or [4]). When $V = \mathbf{P}_n$, then $V^{\triangle}$ becomes the complete flag manifold in $\mathbf{P}_n$ and is denoted by $F(n+1)$. (Cfr. [8]). The Chow ring of $F(n+1)$ can be expressed as

$$A(F(n+1)) = \mathbf{Z}[\gamma_0, \cdots, \gamma_n]$$

subject to the relation

$$\prod_{h=0}^{n} (1 - \gamma_h) = 1 .$$

(*) Nella seduta del 27 novembre 1979.

Using nests $\mathscr{L}$ of linear systems of primals on V and indices $\boldsymbol{k}$, we shall define 'Ehresmann' subvarieties of $V^{\triangle}$ which we shall denote by $[\boldsymbol{k}\,;\,\mathscr{L}\mid V^{\triangle}]$. These are a generalization of subvarieties on tangent direction bundles considered in [7] and [10]. In the special case when $V = \mathbf{P}_n$, these 'Ehresmann' subvarieties correspond with the usual Ehresmann subvarieties in $F(n+1)$ and are denoted by $[\boldsymbol{k}\,;\,F]$. One of the goals of this work is to find a way of determining the cycle classes of these 'Ehresmann' subvarieties of $V^{\triangle}$ in the Chow ring of $V^{\triangle}$. We shall then be able to determine the cycle classes of Jacobian varieties.

In the present Note I, whe shall find the cycle class of an 'Ehresmann' subvariety of $V^{\triangle}$ of codimension one. This is a generalization of the 'invariant lift' of [10], p. 64. Then in the following Note II, we shall prove an intersection formula which enables us to calculate the intersection of any of the 'Ehresmann' subvarieties of $V^{\triangle}$ with one of the 'Ehresmann' subvarieties of codimension one. In the special case when $V = \mathbf{P}_n$, the intersection formula corresponds with the intersection formula proved by Monk in [8] in the complex case. The intersection formula will then enable us in the following Note III to prove the invariance principle which states that the cycle classes of 'Ehresmann' subvarieties of $V^{\triangle}$ in the Chow ring of $V^{\triangle}$ can be determined using a knowledge of the easier corresponding calculus on $F(n+1)$.

An application of this theory is then given to the calculation of the cycle classes of Jacobian subvarieties of V which are defined by an indexed family of nests of linear systems of primals on V. Such Jacobians are projections on V of intersections of 'Ehresmann' subvarieties of $V^{\triangle}$, the cycle classes of which can be computed using the invariance principle. The cycle classes of the Jacobians can then be determined by applying the Gysin homomorphism proved in [5]. The Jacobian subvarieties include the classical Jacobians in its most general form ([11] p. 22, [9]) and the 'generalized Jacobian' of [7] as very special cases. We conclude by giving an explicit formula in a comparatively simple case which is still very much wider than the classical.

Remark. The results of this paper were announced (without proofs) by one of the Authors in [6] in the complex case.

2. 'Ehresmann' subvarieties of $V^{\triangle}$

Ehresmann subvarieties of the flag manifold $F(n+1)$ are defined in terms of permutation symbols (Cfr. [8]). In order to generalize the definition to flag bundles, we use indices and linear systems of primals. Indices are also permutations of the integers $0, 1, \cdots, n$. However, the integers in an index represent codimensions whereas Monk's symbols use actual dimensions. In other words, the index corresponding to a Monk's permutation symbol $(a_0, \cdots, a_n)$ is $(n - a_0, \cdots, n - a_n)$. In this section, we shall give a definition of these 'Ehresmann' subvarieties of $V^{\triangle}$ and determine the cycle class of an 'Ehresmann' subvariety of $V^{\triangle}$ of codimension one.

DEFINITION 2.1. The term *index*, or more precisely (h, n)–index, will mean an $(h+1)$–tuple $\boldsymbol{k} = (k_0, \cdots, k_h)$ of distinct integers with $0 \leq k_i \leq n$ $(i = 0, \cdots, h)$. Let $C(\boldsymbol{k})$ be the set of pairs (i, j) such that

(i) $0 < i \in \{k_0, \cdots, k_j\}$,

(ii) $i - 1 \notin \{k_0, \cdots, k_j\}$,

(iii) $k_j \geq i$,

(iv) $k_{j+1} < i$ or $j = h$.

Then put

$$d_{ij}(\boldsymbol{k}) = |\{0, \cdots, i\} \setminus \{k_0, \cdots, k_j\}|.$$

Remark. We can always extend an (h, n)–index $\boldsymbol{k} = (k_0, \cdots, k_h)$ to an (n, n)–index $\boldsymbol{k}'$ in the following way:

$$\boldsymbol{k}' = (k_0, \cdots, k_h, b_{h+1}, \cdots, b_n)$$

where

$$b_{h+1} < b_{h+2} < \cdots < b_n$$

and

$$\{b_{h+1}, \cdots, b_n\} = \{0, 1, \cdots, n\} \setminus \{k_0, \cdots, k_h\}.$$

Note that $C(\boldsymbol{k}) = C(\boldsymbol{k}')$ and $d_{ij}(\boldsymbol{k}) = d_{ij}(\boldsymbol{k}')$. In this way, one can properly compare indices and permutation symbols of Monk.

DEFINITION 2.2. Relative to a fixed flag

$$E : E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset \mathbf{P}_n \quad , \quad \dim E_i = i ,$$

in $\mathbf{P}_n$, we define an *Ehresmann subvariety* $[\boldsymbol{k}; F]$ of $F = F(n+1)$ for any (h, n)–index $\boldsymbol{k}$, $0 \leq h \leq n$, as consisting of all flags

$$S : S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset \mathbf{P}_n \quad , \quad \dim S_i = i ,$$

satisfying the conditions

$$\dim (S_j \cap E_{n-i}) \geq d_{ij}(\boldsymbol{k}) + j - i , \qquad ((i, j) \in C(\boldsymbol{k})).$$

These Ehresmann subvarieties correspond with Ehresmann subvarieties obtained by Monk's permutation symbols. The cycle class of $[\boldsymbol{k}; F]$ will be denoted by $[\boldsymbol{k}; F]^*$. In the special case where

$$\boldsymbol{k} = (0, 1, \cdots, q-1, q+1),$$

put

$$[\boldsymbol{k}; F]^* = w(q; F).$$

Note that $w(q ; \mathrm{F})$ is the cycle class of a subvariety of $\mathrm{F}(n+1)$ of codimension one.

DEFINITION 2.3. Let

$$\mathscr{L} : \mathscr{L}_1 \subset \mathscr{L}_2 \subset \cdots \subset \mathscr{L}_t \quad , \quad \dim \mathscr{L}_i = i - 1 ,$$

be a nest of linear systems of primals on V. For each $q = 0, \cdots, d$ and any given tangent flag $\mathrm{S} = (\mathrm{S}_0, \mathrm{S}_1, \cdots, \mathrm{S}_d)$ to V, let $\mathscr{L}_i(q, \mathrm{S})$ denote the linear system consisting of those members of $\mathscr{L}_i$ to which S_q is (formally) tangent at S_0. In particular, $\mathscr{L}_i(0, \mathrm{S})$ corresponds to all the members of $\mathscr{L}_i$ through S_0 and $\mathscr{L}_i(d, \mathrm{S})$ to all those with a singularity at S_0. Then corresponding to any (h, t)-index $\boldsymbol{k}$, $0 \leq h \leq d$, we define an '*Ehresmann*' *subvariety* $[\boldsymbol{k} ; \mathscr{L} \mid \mathrm{V}^{\triangle}]$ of $\mathrm{V}^{\triangle}$ to be the subvariety of $\mathrm{V}^{\triangle}$ consisting of all tangent flags satisfying the conditions

$$\dim \mathscr{L}_i(j, \mathrm{S}) \geq d_{ij}(\boldsymbol{k}) - 1 , \qquad ((i, j) \in \mathrm{C}(\boldsymbol{k})) .$$

The cycle class of $[\boldsymbol{k} ; \mathscr{L} \mid \mathrm{V}^{\triangle}]$ will be denoted by $[\boldsymbol{k} ; \mathscr{L} \mid \mathrm{V}^{\triangle}]^*$. As in definition 2.2 above, we also consider the special case, where

$$\boldsymbol{k} = (0, 1, \cdots, q-1, q+1)$$

and we put

$$[\boldsymbol{k} ; \mathscr{L} \mid \mathrm{V}^{\triangle}]^* = w(q ; \mathscr{L} \mid \mathrm{V}^{\triangle}) .$$

Then $w(q ; \mathscr{L} \mid \mathrm{V}^{\triangle})$ is the cycle class of a subvariety of $\mathrm{V}^{\triangle}$ of codimension one.

We shall conclude this section by computing the cycle class $w(q ; \mathscr{L} \mid \mathrm{V}^{\triangle})$ in the Chow ring $\mathrm{A}(\mathrm{V}^{\triangle})$ of $\mathrm{V}^{\triangle}$. It is a generalization of the 'invariant lift' of [10], p. 64. First we shall consider some Ehresmann classes of $\mathrm{F}(n+1)$.

LEMMA 2.4. *Let*

$$\boldsymbol{k} = (0, 1, \cdots, q-r, q-r+2, \cdots, q+1)$$

be a (q, n)*-index. Then the cycle class* $[\boldsymbol{k} ; \mathrm{F}]^*$ *in the Chow ring* $\mathrm{A}(\mathrm{F}(n+1))$ *is given by*

$$[\boldsymbol{k} ; \mathrm{F}]^* = (-1)^r \sigma_r(\gamma_0, \cdots, \gamma_q) .$$

where $\sigma_r(\gamma_0, \cdots, \gamma_q)$ *is the* r-th *elementary symmetric function in* $\gamma_0, \cdots, \gamma_q$.

Proof. Now $[\boldsymbol{k} ; \mathrm{F}]^*$ is the cycle class of the Ehresmann subvariety of $\mathrm{F}(n+1)$ consisting of all flags

$$\mathrm{S} : \mathrm{S}_0 \subset \mathrm{S}_1 \subset \cdots \subset \mathrm{S}_{n-1} \subset \mathbf{P}_n$$

satisfying the single condition

$$\dim(\mathrm{S}_q \cap \mathrm{E}_{n-q+r-2}) \geq r - 1 ,$$

relative to a fixed flag

$$\mathrm{E}:\mathrm{E}_0\subset\mathrm{E}_1\subset\cdots\subset\mathrm{E}_{n-1}\subset\mathbf{P}_n\,.$$

Consider the projection

$$\pi:\mathrm{F}(n+1)\to\mathrm{G}_{q+1}(k^{n+1})\,,\qquad \text{fibre}\quad \mathrm{F}(q+1)\times\mathrm{F}(n-q)\,.$$

Then, from Lemma 1 of [4], π^* is injective and so the Chow ring $\mathrm{A}(\mathrm{G}_{q+1}(k^{n+1}))$ of the Grassmannian $\mathrm{G}_{q+1}(k^{n+1})$ can be considered as a subring of $\mathrm{A}(\mathrm{F}(n+1))$. If λ' is the sub-bundle of the universal bundle over $\mathrm{G}_{q+1}(k^{n+1})$, then we have

$$c_h(\lambda')=(-1)^h\,\sigma_h(\gamma_0,\cdots,\gamma_q)\,,\qquad h=1,\cdots,q+1\,.$$

Using [4] and § 29.3 of [1], we have

$$c_r(\lambda')=[\boldsymbol{k}\,;\mathrm{F}]^*,$$

and the lemma follows.

THEOREM 2.5. (Cfr. 2.4.1 *of* [6]). *For a sufficiently general nest of linear systems*

$$\mathscr{L}:\mathscr{L}_1\subset\mathscr{L}_2\subset\cdots\subset\mathscr{L}_t\,,$$

and for a positive integer $q\le\min(t-1\,,d)$,

$$\begin{aligned}w(q\,;\mathscr{L}\mid\mathrm{V}^{\triangle})&=(q+1)\,w(0\,;\mathscr{L}\mid\mathrm{V}^{\triangle})+\delta_1+\cdots+\delta_q\\&=(q+1)(\rho^*a)+\delta_1+\cdots+\delta_q\,,\end{aligned}$$

where $a\in\mathrm{A}^1(\mathrm{V})$ *is the cycle class of the (unique) member of* $\mathscr{L}_1$ *and* $\rho:\mathrm{V}^{\triangle}\to\mathrm{V}$ *is the natural projection.*

Remark. Note that $w(q\,;\mathscr{L}\mid\mathrm{V}^{\triangle})$ is the cycle class of an 'Ehresmann' subvariety of $\mathrm{V}^{\triangle}$ satisfying the single condition

$$\dim\mathscr{L}_{q+1}(q\,,\mathrm{S})\ge 0\,.$$

Hence $w(q\,;\mathscr{L}\mid\mathrm{V}^{\triangle})$ involves only the single linear system $\mathscr{L}_{q+1}$ (rather than a nest) and it depends only on q and a. (Cfr. [7]).

Proof. Let $f:\mathrm{T}(q\,,d\,;\mathrm{V})\to\mathrm{V}$ be the Grassmannian bundle with fibre $\mathrm{G}_q(k^d)$ associated with the tangent bundle $\mathrm{T}(\mathrm{V})$ of V. Now $g:\mathrm{V}^{\triangle}\to\mathrm{T}(q\,,d\,;\mathrm{V})$ is a fibre bundle, fibre $\mathrm{F}(q)\times\mathrm{F}(d-q)$. From Lemma 1 of [4], g^* is injective and so $\mathrm{A}(\mathrm{T}(q\,,d\,;\mathrm{V}))$ can be considered as a subring of $\mathrm{A}(\mathrm{V}^{\triangle})$. Hence, from Theorem 1, p. 4–19 in [3], the Chow ring $\mathrm{A}(\mathrm{T}(q\,,d\,;\mathrm{V}))$ is generated as a $\rho^*\mathrm{A}(\mathrm{V})$–algebra by the elementary symmetric functions

$$\sigma_i(\delta_1\,,\cdots,\delta_q)\,,\qquad i=1\,,\cdots,q$$

subject to the relation

$$\prod_{i=1}^{d} (1 - \delta_i) = \rho^* c(\mathrm{V}) .$$

If the sub-bundle of the induced bundle $f^* \mathrm{T}(\mathrm{V})$ is denoted by $\xi'(\mathrm{V})$, then taking Chern classes, we have

$$c(\xi'(\mathrm{V})) = \sum_{h=0}^{q} (-1)^h \sigma_h (\delta_1, \cdots, \delta_q) .$$

The inclusion map $i : \mathrm{V} \rightarrow \mathbf{P}_n$ induces the inclusion map

$$\overline{di} : \mathrm{T}(q, d; \mathrm{V}) \rightarrow \mathrm{T}(q, n; \mathbf{P}_n)$$

such that

$$\xi'(\mathrm{V}) = \overline{di}^* \xi'(\mathbf{P}_n) .$$

Hence,

$$-(\delta_1 + \cdots + \delta_q) = c_1(\xi'(\mathrm{V})) = \overline{di}^* c_1(\xi'(\mathbf{P}_n)) . \tag{2.6}$$

Consider the following diagram

$$\begin{tikzcd}
& \mathrm{E} \arrow[dl] \arrow[dr] \arrow[dd, "\mathrm{GL}(q{,}n-q)"] & \\
\mathrm{T}(\mathbf{P}_n) \arrow[dr, "k^q"'] & & \mathrm{F}(n+1) \arrow[dl, "\tau"', "\mathrm{F}(q)\times \mathrm{F}(n-q)"] \\
& \mathrm{T}(q, n; \mathbf{P}_n) &
\end{tikzcd}$$

where $\mathrm{T}(\mathbf{P}_n) \rightarrow \mathrm{T}(q, n; \mathbf{P}_n)$ is $\xi'(\mathbf{P}_n)$ and $\mathrm{E} \rightarrow \mathrm{F}(n+1)$ is a principal bundle β with group and fibre $\Delta(n)$ with $\beta_1, \cdots, \beta_n$ as the corresponding diagonal k^*-bundles in natural order. Thus

$$\tau^* \xi'(\mathbf{P}_n) = \beta_1 \oplus \cdots \oplus \beta_q .$$

From 3.5 of [7],

$$\beta_i = \xi_0^{-1} \otimes \xi_i , \qquad \text{where} \quad c_1(\xi_i) = \gamma_i , \qquad i = 0, \cdots, n .$$

Since τ^* is a monomorphism from Lemma 1 of [4], 2.6 becomes

$$\begin{aligned} -(\delta_1 + \cdots + \delta_q) &= di^* \tau^{*-1} [(-\gamma_0 + \gamma_1) + \cdots + (-\gamma_0 + \gamma_q)] \\ &= di^* \tau^{*-1} [-q\gamma_0 + \gamma_1 + \cdots + \gamma_q] \\ &= di^* \tau^{*-1} [(q+1) w(0; \mathrm{F}) - w(q; \mathrm{F})] \end{aligned} \tag{2.7}$$

(by 2.4.2 of [7]).

Now let $\alpha : \mathrm{W} \to \mathbf{P}_n$ be the flag bundle, fibre $\mathrm{F}(d+1)$, associated with the tangent bundle $\mathrm{T}(\mathbf{P}_n)$ of $\mathbf{P}_n$.

Also $\varphi : \mathrm{F}(n+1) \to \mathrm{W}$ is a fibre bundle, fibre $\mathrm{F}(n-d)$. From Lemma 1 of [4], φ^* is injective and so $\mathrm{A}(\mathrm{W})$ can be considered as a subring of $\mathrm{A}(\mathrm{F}(n+1))$. Since $\mathrm{V} \to \mathbf{P}_n$ is an inclusion, then we have the natural injection

$$\theta : \mathrm{V}^{\triangle} \to \mathrm{W}.$$

Thus 2.7 implies that

$$\delta_j = \theta^*(\gamma_0 - \gamma_j).$$

Hence

$$\theta^*(-\gamma_0) = \theta^* \alpha^*(-\gamma_0) = \rho^* i^*(-\gamma_0)$$

and

$$\theta^*(-\gamma_j) = \rho^* i^*(-\gamma_0) + \delta_j, \qquad j = 1, \cdots, q. \tag{2.8}$$

Let

$$\mathscr{L}_2^{(1)}, \cdots, \mathscr{L}_2^{(d-q)}$$

be $d-q$ pencils of prime sections. The condition

$$\dim \mathscr{L}_2^{(i)}(q, \mathrm{S}) \geq 0$$

defines the cycle class $[\boldsymbol{k} ; \mathscr{L}^{(i)} \mid \mathrm{V}^{\triangle}]^*$ where

$$\boldsymbol{k} = (0, 2, \cdots, q+1).$$

Then since $[\boldsymbol{k} ; \mathrm{F}]^* \in \mathrm{A}^q(\mathrm{W})$, we have that

$$\begin{aligned}
(2.9) \quad [\boldsymbol{k} ; \mathscr{L}^{(i)} \mid \mathrm{V}^{\triangle}]^* &= \theta^*([\boldsymbol{k} ; \mathrm{F}]^*) \\
&= \theta^*((-1)^q \sigma_q(\gamma_0, \cdots, \gamma_q)), \qquad \text{by Lemma 2.4 if } r = q \\
&= \theta^*(\sigma_q(-\gamma_0, \cdots, -\gamma_q)) \\
&= \sigma_q(\rho^* b, \delta_1 + \rho^* b, \cdots, \delta_q + \rho^* b), \qquad \text{from 2.8}
\end{aligned}$$

where $i^*(-\gamma_0) = b \in \mathrm{A}^1(\mathrm{V})$ is the cycle class of the (unique) member of $\mathscr{L}_1^{(i)}$. Since $w(q ; \mathscr{L} \mid \mathrm{V}^{\triangle}) \in \mathrm{A}(\mathrm{T}(q, d ; \mathrm{V}))$, a subring of $\mathrm{A}(\mathrm{V}^{\triangle})$, it follows that we can write

$$w(q ; \mathscr{L} \mid \mathrm{V}^{\triangle}) = \rho^* y + n(\delta_1 + \cdots + \delta_q)$$

where n is an integer and $y \in \mathrm{A}^1(\mathrm{V})$. Now from 2.7, we have

$$w(q ; \mathscr{L}' \mid \mathrm{V}^{\triangle}) = (q+1)\rho^* b + \delta_1 + \cdots + \delta_q \tag{2.10}$$

where $\mathscr{L}'$ is a linear system of prime sections. Since the intersection of $w(q ; \mathscr{L} \mid \mathrm{V}^{\triangle})$ with a general fibre is the same for all $\mathscr{L}$, n must be the same for all $\mathscr{L}$. Hence in view of 2.10.

$$w(q ; \mathscr{L} \mid \mathrm{V}^{\triangle}) = \rho^* y + \delta_1 + \cdots + \delta_q. \tag{2.11}$$

Thus the conditions

$$\dim \mathscr{L}_2^{(i)}(q\,,\mathrm{S}) \geq 0\,, \qquad i = 1\,,\cdots,d-q$$

and

$$\dim \mathscr{L}_{q+1}(q\,,\mathrm{S}) \geq 0$$

define the cycle class $j^{\triangle}$ of $\mathrm{V}^{\triangle}$ given by

$$j^{\triangle} = w\,(q\,;\mathscr{L}\mid \mathrm{V}^{\triangle})\cdot\prod_{i=1}^{d-q}[\boldsymbol{k}\,;\mathscr{L}^{(i)}\mid \mathrm{V}^{\triangle}]^{*}$$

$$= (\rho^{*}y+\delta_1+\cdots+\delta_q)\,[\sigma_q(\rho^{*}b\,,\delta_1+\rho^{*}b\,,\cdots,\delta_q+\rho^{*}b)]^{d-q},$$

by 2.9, 2.11.

Hence from the Gysin homomorphism stated as Theorem 2.2.1 in [6] for the complex case and proved for the general case in [5], we have

$$\mathrm{P}\,(x_1\,,\cdots,x_q) = (y+x_1+\cdots+x_q)\,[\sigma_q(b\,,x_1+b\,,\cdots,x_q+b)]^{d-q}$$

$$\hat{\mathrm{P}}\,(x_1\,,\cdots,x_q) = (x_1\,,\cdots,x_q)^{d-q+1}\,\mathrm{P}\left(\frac{1}{x_1},\cdots,\frac{1}{x_q}\right)$$

$$= [y\sigma_q(\boldsymbol{x})+\sigma_{q-1}(\boldsymbol{x})]\,[1+2\,b\sigma_1(\boldsymbol{x})+3\,b^2\,\sigma_2(\boldsymbol{x})+\cdots+(q+1)\,b^q\,\sigma_q(\boldsymbol{x})]^{d-q}.$$

The cycle class $j = \rho_{*}(j^{\triangle})$ of the corresponding Jacobian J is equal to the coefficient of

$$x_1^d\,x_2^{d-1},\cdots,x_q^{d-q+1}$$

in

$$\prod_{i=1}^{q}(1+d_1\,x_h+\cdots+d_d\,x_h^d)\prod_{i=h+1}^{q}(x_h-x_i)\,(x_1\,,\cdots,x_q)^{d-q}\,[y\sigma_q(\boldsymbol{x})+$$

$$+\sigma_{q-1}(\boldsymbol{x})]\,[1+2\,b\sigma_1(\boldsymbol{x})+\cdots+(q+1)\,b^q\,\sigma_q(\boldsymbol{x})]^{d-q}$$

i.e.
$$j = \rho_{*}(j^{\triangle}) = d_1+y+2\,(d-q)\,b\,.$$

By Severi (p. 21 of [11]), since the sum of the dimensions of

$$\mathscr{L}_{q+1}\,,\mathscr{L}_2^{(1)},\cdots,\mathscr{L}_2^{(d-q)}$$

is equal to d and so the Jacobian

$$\mathrm{J}\,[\mathscr{L}\,;\mathscr{L}^{(1)},\cdots,\mathscr{L}^{(d-q)}] = \mathrm{X}_{d-1}+(q+1)\,\mathrm{A}+2\,(\mathrm{B}_1+\cdots+\mathrm{B}_{d-q})$$

i.e.
$$j = d_1+(q+1)\,a+2\,(d-q)\,b\,.$$

Comparing the two values of j, we obtain

$$y = (q+1)\,a\,.$$

Hence

$$w\,(q\,;\mathscr{L}\mid \mathrm{V}^{\triangle}) = (q+1)\,(\rho^{*}a)+\delta_1+\cdots+\delta_q\,.$$

This completes the proof of the theorem.