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On an inequality related to the motion, in any dimension, of viscous, incompressible fluids

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Analisi matematica. — *On an inequality related to the motion, in any dimension, of viscous, incompressible fluids.* Nota II [(*)] di
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RIASSUNTO. — Si dimostrano teoremi di esistenza, unicità e dipendenza continua per le soluzioni debole e forte della disequazione di Navier–Stokes considerata nella Nota I.

3. – We now prove existence, uniqueness and continuous dependence theorems for the weak and strong solutions of the Navier-Stokes inequality satisfying the initial and boundary conditions (2.2) and (2.3).

THEOREM 1. *If* $\vec{f}(t) \in L^2(0, T; (N^1)')$, $\vec{\alpha} \in N^0$, *there exists at least one function* $\vec{v}(t)$ *satisfying conditions* $a_2')$, $b_2')$ *(weak solution).*

Let β be a penalisation operator relative to the convex set K defined in § 2; denoting by P the projection operator on K we can set (see, for instance, [1], ch. 2, § 2.2)

$$\beta(\vec{z}) = \vec{z} - P\vec{z} \tag{3.1}$$

It is well known that β is a monotone, hemicontinuous operator from $L^2(0, T; N^0)$ to itself. Denoting by s a number $> m/2$, let moreover $\{\vec{g}_j\}$ be a basis in N^s.

Setting

$$\vec{v}_n(t) = \sum_{j=1}^{n} \alpha_{jn}(t)\,\vec{g}_j \quad , \quad \vec{f}_n(t) = \Pi_n \vec{f}(t)\ ^{(1)}, \tag{3.2}$$

consider the system of ordinary differential equations

$$(\vec{v}_n'(t) - \mu\Delta\vec{v}_n(t) + n\beta(\vec{v}_n(t)) - \vec{f}_n(t), \vec{g}_j)_{N^0} + b(\vec{v}_n(t), \vec{v}_n(t)\,\vec{g}_j) = 0 \qquad (j = 1, \cdots, n) \tag{3.3}$$

with the initial conditions

$$\vec{v}_n(0) = \Pi_n \vec{\alpha} = \vec{\alpha}_n \tag{3.4}$$

By well known theorems, (3.3), (3.4) admit, $\forall n$, a solution for sufficiently small t.

(*) Pervenuta all'Accademia il 29 agosto 1979.
(1) Π_n denotes the projection operator on the subspace defined by $\vec{g}_1, \vec{g}_2, \cdots, \vec{g}_n$.

In order to obtain some a priori estimates for $\vec{v}_n(t)$ from which, in particular, will follow the existence of such a solution on the whole of $[0, T]$, we multiply (3.3) by $\alpha_{jn}(t)$, add and integrate over $[0, t]$ $(0 < t \leq T)$; we then obtain

$$\begin{aligned}(3.5)\qquad \tfrac{1}{2}\|\vec{v}_n(t)\|^2_{N^0} + \int_0^t \{\mu \|\vec{v}_n(\eta)\|^2_{N^1} + b(\vec{v}_n(\eta), \vec{v}_n(\eta), \vec{v}_n(\eta)) + \\ + n(\beta(\vec{v}_n(\eta)), \vec{v}_n(\eta))_{N^0} - (\vec{f}_n(\eta), \vec{v}_n(\eta))_{N^0}\} \, d\eta = \tfrac{1}{2}\|\Pi_n \vec{\alpha}\|^2_{N^0}\end{aligned}$$

Since $(\beta(\vec{v}_n), \vec{v}_n)_{N^0} \geq 0$ and $b(\vec{v}_n, \vec{v}_n, \vec{v}_n) = 0$, from (3.5) it follows that

$$(3.6)\qquad \|\vec{v}_n(t)\|_{N^0} \leq M_1 \quad , \quad \int_0^T \|\vec{v}_n(t)\|^2_{N^1} \, dt \leq M_2 ,$$

with M_i independent of n. By (3.6) the solution $\vec{v}_n(t)$ of (3.3) exists on the whole of $[0, T]$; it is, moreover, possible to select from $\{\vec{v}_n\}$ a subsequence (again denoted by $\{\vec{v}_n\}$) such that

$$(3.7)\qquad \lim_{n\to\infty} \vec{v}_n(t) \underset{L^2(0,T;N^1)}{=} \vec{v}(t) \quad , \quad \lim_{n\to\infty} \vec{v}_n(t) \underset{L^\infty(0,T;N^0)}{=} \vec{v}(t)$$

respectively in the weak and weak-star topologies.

On the other hand, by (3.5), (3.6),

$$(3.8)\qquad \int_0^T (\beta(\vec{v}_n), \vec{v}_n)_{N^0} \, dt \leq M_3/n$$

and, consequently,

$$(3.9)\qquad \lim_{n\to\infty} \int_0^T (\beta(\vec{v}_n), \vec{v}_n)_{N^0} \, dt = 0 .$$

Hence,

$$(3.10)\qquad \|\beta(\vec{v})\|_{L^2(0,T;N^0)} = 0 \Rightarrow \vec{v}(t) \in L^2(0, T; K)$$

By (3.7), (3.10) the limit function $\vec{v}(t)$ satisfies $a_2')$.

Let $\vec{\psi}$ be any function $\in H_0^1(0, T; N^s)$ and set $\vec{\psi}_n = \Pi_n \vec{\psi}$; assuming (as is obviously possible) that the $\vec{g}_j$ are orthonormal and bearing in mind that when $s > m/2$ the embedding of N^s in L^∞ is continuous, it follows from (3.3) that

$$(3.11) \qquad \left| \int_0^T (\vec{v}_n', \vec{\psi})_{N^0} \, dt \right| = \left| \int_0^T (\vec{v}_n', \vec{\psi}_n)_{N^0} \, dt \right| =$$

$$= \left| \int_0^T \{-\mu (\vec{v}_n, \vec{\psi}_n)_{N^1} - n (\beta(\vec{v}_n), \vec{\psi}_n)_{N^0} - b(\vec{v}_n, \vec{v}_n, \vec{\psi}_n) + (\vec{f}_n, \vec{\psi})_{N^0}\} \, dt \right| \leq$$

$$\leq \mu \| \vec{v}_n \|_{L^2(0,T;N^1)} \| \vec{\psi}_n \|_{L^2(0,T;N^1)} + \| \vec{f}_n \|_{L^2(0,T;(N^1)')} \| \vec{\psi}_n \|_{L^2(0,T;N^1)} +$$

$$+ \| \vec{v}_n \|_{L^\infty(0,T;N^0)} \| \vec{v}_n \|_{L^2(0,T;N^1)} \| \vec{\psi}_n \|_{L^2(0,T;N^s)} + \left| n \int_0^T (\beta(\vec{v}_n), \vec{\psi}_n)_{N^0} \, dt \right| .$$

Observe, on the other hand that, by (3.8), since $\beta(\vec{v}_n) = 0$ when $|\vec{v}_n| \leq c$ and, by the definition of K, $\beta(\vec{v}_n) \cdot \vec{v}_n = |\beta(\vec{v}_n)| \, |\vec{v}_n|$,

$$(3.12) \qquad M_3 \geq \int_0^T n (\beta(\vec{v}_n), \vec{v}_n)_{N^0} \, dt \geq nc \int_Q |\beta(\vec{v}_n)| \, dQ .$$

Hence, substituting (3.12) into (3.11), we obtain, by (3.6), $\forall \varepsilon > 0$,

$$(3.13) \qquad \left| \int_0^T (\vec{v}_n', \vec{\psi})_{N^0} \, dt \right| \leq M_4 \| \vec{\psi} \|_{L^2(0,T;N^s)} + (M_3/c) \| \vec{\psi} \|_{L^\infty(0,T;L^\infty)} \leq$$

$$\leq M_5 \| \vec{\psi} \|_{H_0^{\frac{1}{2}+\varepsilon}(0,T;N^s)}$$

and, consequently, since $H_0^1(0, T; N^s)$ is dense in $H_0^{\frac{1}{2}+\varepsilon}(0, T; N^s)$, denoting by $(N^s)'$ the dual of N^s,

$$(3.14) \qquad \| \vec{v}_n'(t) \|_{H^{-\frac{1}{2}-\varepsilon}(0,T;(N^s)')} \leq M_5 .$$

Since the embedding of $L^2(0, T; N^1) \cap H^{\frac{1}{2}-\varepsilon}(0, T; (N^s)')$ into $L^2(0, T; N^0)$ is completely continuous, we have, by (3.14),

$$(3.15) \qquad \lim_{n \to \infty} \vec{v}_n(x, t) = \vec{v}(x, t) \qquad \text{a.e. in } Q.$$

Let $\vec{\varphi}(t)$ be an arbitrary function $\in L^2(0, T; N^s)$, with $\vec{\varphi}'(t) \in L^2(0, T; N^s)$, $|\vec{\psi}(x, t)| < c$; setting

$$(3.16) \qquad \vec{\varphi}(t) = \sum_{j=1}^{\infty} \gamma_j(t) \vec{g}_j \quad , \quad \vec{\varphi}_p(t) = \sum_{j=1}^{p} \gamma_j(t) \vec{g}_j ,$$

it is obvious, since the embedding of $H^1(0, T; N^s)$ in $C^0(\bar{Q})$ is completely continuous, that, when $p \geq \bar{p}$ sufficiently large, $|\vec{\varphi}_p| \leq c$.

Assuming then that $p \geq \bar{p}$ and setting $\sigma_j = \gamma_j$ when $j \leq p$, $\sigma_j = 0$ when $j > p$, let us multiply (3.3) by $\alpha_{jn}(t) - \sigma_j(t)$; taking $n \geq p$ and adding, we obtain

$$(3.17) \qquad (\vec{v}_n'(t), \vec{v}_n(t) - \vec{\varphi}_p(t))_{N^0} + \mu(\vec{v}_n(t), \vec{v}_n(t) - \vec{\varphi}_p(t))_{N^1} + \\ + n(\beta(\vec{v}_n(t)), \vec{v}_n(t) - \vec{\varphi}_p(t))_{N^0} + b(\vec{v}_n(t), \vec{v}_n(t), \vec{v}_n(t) - \vec{\varphi}_p(t)) - \\ - (\vec{f}_n(t), \vec{v}_n(t) - \vec{\varphi}_p(t))_{N^0} = 0.$$

Observe now that, since $|\vec{\varphi}_p| \leq c$ (hence $\beta(\vec{\varphi}_p) = 0$) we have

$$(3.18) \qquad (\beta(\vec{v}_n), \vec{v}_n - \vec{\varphi}_p)_{N^0} = (\beta(\vec{v}_n) - \beta(\vec{\varphi}_p), \vec{v}_n - \vec{\varphi}_p)_{N^0} \geq 0.$$

Consequently, denoting by $\psi(t)$ any function $\in C^0[0, T]$, with $\psi(t) \geq 0$, integrating (3.17) between 0 and $t \in [0, T]$ and applying Green's formula, we obtain

$$(3.19) \qquad \int_0^T \psi(t) \{ \tfrac{1}{2} \| \vec{v}_n(t) - \vec{\varphi}_p(t) \|^2_{N^0} - \tfrac{1}{2} \| \Pi_n \vec{\alpha} - \vec{\varphi}_p(0) \|^2_{N^0} + \\ + \int_0^t [(\vec{\varphi}_p'(\eta), \vec{v}_n(\eta) - \vec{\varphi}_p(\eta))_{N^0} + \mu(\vec{v}_n(\eta), \vec{v}_n(\eta) - \vec{\varphi}_p(\eta))_{N^1} + \\ + b(\vec{v}_n(\eta), \vec{v}_n(\eta), v_n(\eta) - \varphi_p(\eta)) - (\vec{f}_n(\eta), \vec{v}_n(\eta) - \vec{\varphi}_p(\eta))_{N^0}]\, d\eta \}\, dt \leq 0.$$

Let $n \to \infty$ in (3.19); the function $\vec{v}(t)$ satisfies, by (3.7), (3.15) and well known properties of the weak limit, the relation

$$(3.20) \qquad \int_0^T \psi(t) \{ \tfrac{1}{2} \| \vec{v}(t) - \vec{\psi}_p(t) \|^2_{N^0} - \tfrac{1}{2} \| \vec{\alpha} - \vec{\varphi}_p(0) \|^2_{N^0} + \\ + \int_0^t [(\vec{\varphi}_p'(\eta), \vec{v}(\eta) - \vec{\varphi}_p(\eta))_{N^0} + \mu(\vec{v}(\eta), \vec{v}(\eta) - \vec{\varphi}_p(\eta))_{N^1} + \\ + b(\vec{v}(\eta), \vec{v}(\eta), \vec{v}(\eta) - \vec{\varphi}_p(\eta)) - \langle \vec{f}(\eta), \vec{v}(\eta) - \vec{\varphi}_p(\eta) \rangle]\, d\eta \}\, dt \leq 0.$$

Hence, $\vec{v}(t)$ satisfies b_2') $\forall$ function $\vec{\varphi}_p(t)$ defined by (3.16); letting $p \to \infty$ and observing that the class of functions $\vec{\varphi}$ defined by (3.16) is dense in that of the test functions considered in b_2'), we can conclude that $\vec{v}(t)$ satisfies b_2'). The theorem is therefore proved.

THEOREM 2. *The weak solution given in Theorem 1 is unique and depends continuously on* $\vec{f}$ *and* $\vec{\alpha}$.

Assume, in fact, that there exists two functions, $\vec{u}(t)$ and $\vec{v}(t)$, satisfying a_2'), b_2') and set $\vec{w}(t) = \frac{1}{2}(\vec{u}(t) + \vec{v}(t))$. Denote, moreover, by $\{\vec{w}_j\}$ a regularising sequence associated to $\vec{w}$ (see note 1, p. 4).

Writing condition b_2') for $\vec{u}(t)$ and $\vec{v}(t)$, setting $\vec{\varphi} = \vec{w}_j$ and adding we obtain

$$\begin{aligned}(3.21)\qquad & \tfrac{1}{2}\|\vec{v}(t) - \vec{w}_j(t)\|^2_{N^0} + \tfrac{1}{2}\|\vec{u}(t) - \vec{w}_j(t)\|^2_{N^0} + \\ & + \int_0^t \{2(\vec{w}_j'(\eta), \vec{w}(\eta) - \vec{w}_j(\eta))_{N^0} + \mu(\vec{v}(\eta), \vec{v}(\eta) - \vec{w}_j(\eta))_{N^1} + \\ & + \mu(\vec{u}(\eta), \vec{u}(\eta) - \vec{w}_j(\eta))_{N^1} + b(\vec{v}(\eta), \vec{v}(\eta), \vec{v}(\eta) - \vec{w}_j(\eta)) + \\ & + b(\vec{u}(\eta), \vec{u}(\eta), \vec{u}(\eta) - \vec{w}_j(\eta)) - (\vec{f}(\eta), \vec{w}(\eta) - \vec{w}_j(\eta))_{N^0}\}\, d\eta \le 0 .\end{aligned}$$

Hence, by (2.6)

$$\begin{aligned}(3.22)\qquad & \tfrac{1}{2}\|\vec{v}(t) - \vec{w}_j(t)\|^2_{N^0} + \tfrac{1}{2}\|\vec{u}(t) - \vec{w}_j(t)\|^2_{N^0} + \\ & + \int_0^t \{\mu(\vec{v}(\eta), \vec{v}(\eta) - \vec{w}_j(\eta))_{N^1} + \mu(\vec{u}(\eta), \vec{u}(\eta) - \vec{w}_j(\eta))_{N^1} + \\ & + b(\vec{v}(\eta), \vec{v}(\eta), \vec{v}(\eta) - \vec{w}_j(\eta)) + b(\vec{u}(\eta), \vec{u}(\eta), \vec{u}(\eta) - \vec{w}_j(\eta)) - \\ & - (\vec{f}(\eta), \vec{w}(\eta) - \vec{w}_j(\eta))_{N^0}\}\, d\eta \le 0\end{aligned}$$

and, letting $j \to \infty$,

$$\begin{aligned}(3.23)\qquad & \left\| \frac{\vec{u}(t) - \vec{v}(t)}{2} \right\|^2_{N^0} + \int_0^t \Big\{ \frac{\mu}{2} \|\vec{u}(\eta) - \vec{v}(\eta)\|^2_{N^1} + \\ & + \tfrac{1}{2} b(\vec{v}(\eta), \vec{v}(\eta), \vec{v}(\eta) - \vec{u}(\eta)) + \tfrac{1}{2} b(\vec{u}(\eta, \vec{u}(\eta), \vec{u}(\eta) - \vec{v}(\eta)) \Big\}\, d\eta \le 0.\end{aligned}$$

On the other hand, since $b(\vec{w}, \vec{v}, \vec{w}) = -b(\vec{w}, \vec{w}, \vec{v})$, setting $\vec{z} = \vec{u} - \vec{v}$,

$$\begin{aligned}(3.23)\qquad & |b(\vec{u}, \vec{u}, \vec{u} - \vec{v}) - b(\vec{v}, \vec{v}, \vec{u} - \vec{v})| = |b(\vec{u}, \vec{z}, \vec{z}) + b(\vec{z}, \vec{v}, \vec{z})| = \\ & = |b(\vec{z}, \vec{v}, \vec{z})| \le c\|\vec{z}\|_{N^1}\|\vec{z}\|_{N^0} \le c\,[\varepsilon\|\vec{z}\|^2_{N^1} + \tau_\varepsilon\|\vec{z}\|^2_{N^0}] .\end{aligned}$$

Introducing (3.23) into (3.22) and choosing $\varepsilon < \mu/c$, we conclude that it must necessarily be $\vec{u} = \vec{v}$. The uniqueness theorem is therefore proved.

By exactly the same procedure we can show that the solution depends continuosly on $\vec{f}$ and $\vec{\alpha}$, i.e. that

$$\begin{aligned}(3.24)\qquad & \|\vec{f}_n(t) - \vec{f}(t)\|_{L^2(0,T);(N^1)')} \to 0 , \|\vec{\alpha}_n - \vec{\alpha}\|_{N^0} \to 0 , \\ & \Rightarrow \|\vec{v}_n(t) - \vec{v}(t)\|_{L^\infty(0,T;N^0)\cap L^2(0,T;N^1)} \to 0 .\end{aligned}$$

THEOREM 3. *If* $f(t) \in L^2(0, T; L^2)$, $\vec{\alpha} \in N^1$ *and* Ω *is sufficiently smooth then there exists a unique function* $\vec{v}(t)$ *satisfying conditions* a$_2''$), b$_2''$).

The theorem will obviously be proved if we show that the weak solution $\vec{v}(t)$ given in Theorems 1 and 2 is such that $\Delta\vec{v}(t) \in L^2(0, T; N^0)$.

Let us choose a " special " basis $\{\vec{g}_j\}$, constituted by the eigenfunctions in N^1 of the operator Δ and denote by λ_j the corresponding eigenvalues. By the smoothness assumptions made on Ω, these eigenfunctions constitute a basis in N^s. We shall therefore assume, from now on, that

$$(3.25) \qquad (\vec{g}_j, \vec{\varphi})_{N^1} = \lambda_j (\vec{g}_j, \vec{\varphi})_{N^0} \, \forall \vec{\varphi} \in N_1, \vec{g}_j \in N^s, (\vec{g}_j, \vec{g}_k)_{N^0} = \delta_{jk}.$$

Setting

$$(3.26) \qquad \vec{z}_n(t) = \sum_{j=1}^{n} \gamma_{jn}(t) \vec{g}_j,$$

consider the system of linear equations

$$(3.27) \qquad (\vec{z}_n'(t) - \mu\Delta\vec{z}_n(t) + n\beta(\vec{z}_n(t)) - \vec{f}_n(t), \vec{g}_j)_{N^0} + b(\vec{v}(t), \vec{v}(t), \vec{g}_j) = 0 \qquad (j = 1, \cdots, n),$$

with the initial conditions

$$(3.28) \qquad \vec{z}_n(0) = \Pi_n \vec{\alpha} = \vec{\alpha}_n$$

where $\vec{v}(t)$ is the weak solution given in Theorems 1 and 2.

By exactly the same procedure followed in Theorem 1 it can be proved that the sequence $\{\vec{z}_n(t)\}$ converges to a function $\vec{z}(t)$ satisfying a$_2'$) and the inequality

$$(3.29) \qquad \tfrac{1}{2}\| \vec{z}(t) - \vec{\varphi}(t) \|_{N^0}^2 + \int_0^t \{(\vec{\varphi}', \vec{z} - \vec{\varphi})_{N^0} + \mu(\vec{z}, \vec{z} - \vec{\varphi})_{N^1} - (\vec{f}, \vec{z} - \vec{\varphi})_{N^0} + b(\vec{v}, \vec{v}, \vec{z} - \vec{\varphi})\} \, d\eta - \tfrac{1}{2}\| \vec{\alpha} - \vec{\varphi}(0) \|_{N^0}^2 \leq 0$$

$\forall \vec{\varphi}(t) \in L^2(0, T; K \cap N^1)$ with $\vec{\varphi}'(t) \in L^2(0, T; N^0)$.

It follows, by a uniqueness theorem analogous to Theorem 2, that

$$(3.30) \qquad \vec{z}(x, t) = \vec{v}(x, t).$$

Let us now multiply (3.27) by $-\lambda_j \alpha_{jn}(t)$ add and integrate over $[0, t]$, $t \in [0, T]$; we obtain

$$(3.31) \qquad \tfrac{1}{2}\| \vec{z}_n(t) \|_{N^1}^2 + \int_0^t \{\mu \| \Delta\vec{z}_n(\eta) \|_{N^0}^2 - b(\vec{v}(\eta), \vec{v}(\eta), \Delta\vec{z}_n(\eta)) - (n\beta(\vec{z}_n(\eta)), \Delta\vec{z}_n(\eta))_{N^0} - (\vec{f}_n(\eta), \Delta\vec{z}_n(\eta))_{N^0}\} \, d\eta = \tfrac{1}{2}\| \Pi_n \vec{\alpha} \|_{N^1}^2.$$

Observe now that

$$(3.32) \qquad -(\beta(z_n), \Delta z_n)_{N^0} = \sum_{j=1}^{m} \left(\frac{\partial}{\partial x_j} \beta(\vec{z}_n), \frac{\partial \vec{z}_n}{\partial x_j} \right)_{N^0}$$

and, by the monotonicity of β, setting $\vec{z}_n(x, t) = 0$ when $x \notin \overline{\Omega}$,

$$(3.33) \qquad \int_{\Omega} (\beta(\vec{z}_n(x+h, t)) - \beta(\vec{z}_n(x, t)) \cdot (\vec{z}_n(x+h, t) - \vec{z}_n(x, t)) \, d\Omega \geq 0 .$$

Since β satisfies a Lipschitz condition (N^1 being a Hilbert space) it follows from (3.22), (3.33) that

$$(3.34) \qquad -(\beta(\vec{z}_n), \Delta\vec{z}_n)_{N^0} \geq 0 .$$

On the other hand, since $|\vec{v}(x, t)| \leq c$ and $\vec{v}(t) \in L^2(0, T; N^1)$,

$$(3.35) \qquad \int_0^T |b(\vec{v}, \vec{v}, \Delta\vec{z}_n)| \, d\eta \leq c \sqrt{M_2} \, \| \Delta\vec{z}_n \|_{L^2(0,T;N^0)}$$

Hence, by (3.31), (3.34), (3.35)

$$(3.36) \qquad \| \vec{z}_n(t) \|_{N^1} \leq M_6 \quad , \quad \int_0^T \| \Delta\vec{z}_n(t) \|_{N^0}^2 \, dt \leq M_7$$

and, consequently,

$$(3.37) \qquad \vec{z}(t) \in L^\infty(0, T; N^1) \quad , \quad \Delta\vec{z}(t) \in L^2(0, T; N^0) .$$

From (3.29), (3.30), (3.37) it follows that $\vec{z}(t)$ satisfies conditions $a_2'')$, $b_2'')$; this completes the proof of the Theorem.