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Asymptotic stability properties for nonlinear diffusion Volterra equations

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Equazioni funzionali. — *Asymptotic stability properties for non-linear diffusion Volterra equations.* Nota (*) di ANDREA SCHIAFFINO e ALBERTO TESEI, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Si dimostra la stabilità asintotica uniforme dell'unico equilibrio non banale di un'equazione integrodifferenziale di Volterra con diffusione, soggetta a condizioni ai limiti di tipo Dirichlet.

1. INTRODUCTION

In the present paper we study Liapunof stability properties of the equilibrium solutions to the integro-partial differential problem:

$$
(1)\qquad \begin{cases} \partial_t u(t\,,x) = \mathrm{D}_2 u(t\,,x) - u(t\,,x)\displaystyle\int_{-\infty}^{t} \mathrm{d}s\, k_0(t-s\,,x)\, u(s\,,x) - b_0(x)\, u^2(t\,,x) & \text{in } (0\,,+\infty)\times\Omega \\ u(t\,,x) = 0 & \text{on } (0\,,+\infty)\times\partial\Omega \\ u(t\,,x) = u_0(t\,,x) & \text{in } (-\infty\,,0]\times\Omega, \end{cases}
$$

where $\Omega \subset \mathrm{R}^{\mathrm{d}}$ ($\mathrm{d} \leq 3$) is an open bounded domain with smooth boundary $\partial\Omega$, $(\mathrm{D}_2\, u)(x) = \sum_{i,j=1}^{\mathrm{d}} \partial_i\,(a_{ij}(x)\,\partial_j\, u(x)) + a(x)\, u(x)\ (x \in \Omega)$ defines a linear, formally self-adjoint elliptic operator of the second order, and $b_0\,, k_0\,, u_0$ are given nonnegative functions. We can think of the integro-differential equation in problem (1) as of a generalization of Volterra's population equation with infinite delay including space dependent effects [10]. For simplicity, only the case of homogeneous Dirichlet boundary conditions will be considered; however, more general boundary conditions can be dealt with similarly.

In the space-clamp case, namely

$$
(2)\qquad \begin{cases} u'(t) = \mu u(t) - u(t)\displaystyle\int_{-\infty}^{t} \mathrm{d}s\, k(t-s)\, u(s) - bu^2(t) & (t > 0) \\ u(t) = u_0(t) & (t \leq 0), \end{cases}
$$

(*) Pervenuta all'Accademia l'1 agosto 1979.

where $k(\cdot) \in L^1(0, +\infty)$ and $u_0(\cdot)$ is continuous and bounded on $(-\infty, 0]$, the asymptotical stability of the nontrivial equilibrium $\hat{u} = \mu/(|k|_{L^1} + b)$ (here $\mu > 0$) of Volterra's equation follows from the spectral condition $[\zeta + \hat{u}k^*(\zeta) + b\hat{u}] \neq 0$ for any $\zeta \in C$, $Re\, \zeta \geq 0$ ($k^*(\cdot)$ denoting the Laplace transform of $k(\cdot)$). A generalization of such result for a more particular version of the Volterra equation in (1), endowed with homogeneous Neumann boundary conditions, was proved in [9]; in the same frame, conditions ensuring the global attractivity (in the supremum norm) of the unique nontrivial equilibrium with respect to strictly positive initial data were given in [5].

In the present paper a linearization argument is used to investigate the stability of the unique strictly positive equilibrium of the equation in (1) by means of a characteristic equation, thus generalizing to the present case the above referred spectral condition. A major difficulty with respect to [9] is the noncommutativity of the terms in the characteristic equation, to be overcome by a slight refinement of the results in [6]; regularity results are also needed, which give rise to the requirement $d \leq 3$.

2. Statement of the results

Let Ω be an open bounded domain of R^d ($d \leq 3$) with smooth boundary $\partial\Omega$. We shall denote by X (norm $|\cdot|_X$) the Banach space $C_0(\Omega)$ of continuous real functions on $\bar{\Omega} := \Omega \cup \partial\Omega$ which vanish on $\partial\Omega$, endowed with the supremum norm. The algebra $\mathscr{L}(X)$ of linear bounded operators on X will be thought of as endowed with the operatorial norm $\|\cdot\|$. We shall also be dealing with the Banach space $L^1(0, +\infty; \mathscr{L}(X))$, $C^k(\bar{\Omega})$ ($k \geq 0$; $C^0(\bar{\Omega}) = : C(\bar{\Omega})$) and the Sobolev spaces $H^2(\Omega)$, $H_0^1(\Omega)$ [2]. For any $f \in L^1(0, + \infty; \mathscr{L}(X))$ we shall denote by $f^*(\cdot)$ the corresponding Laplace transform.

Let us consider the following operator A_0 with dense domain in X:

$$\begin{cases} D(A_0) := \{u \in X \mid D_2 u \in X\} \\ (A_0 u)(\cdot) := -(D_2 u)(\cdot) := -\sum_{i,j=1}^{d} \partial_i (a_{ij}(\cdot)(\partial_j u)(\cdot)) - a(\cdot)\, u(\cdot) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (u \in D(A_0)). \end{cases}$$

We shall assume $a_{ij}(\cdot) = a_{ji}(\cdot) \in C^1(\bar{\Omega})$ $(i, j = 1, \cdots, d)$ and $a(\cdot) \in C(\bar{\Omega})$, so that A_0 is densely defined and formally self-adjoint. Moreover, the uniform ellipticity of $-A_0$ will be required; thus we shall denote by λ the principal eingevalue of $-A_0$ and by χ the corresponding positive eigenfunction:

$$A_0 \chi + \lambda\chi = 0 \quad , \quad \chi \in H_0^1(\Omega) \quad , \quad \int_\Omega \chi^2(x)\, dx = 1 .$$

Let us remark that $\chi \in X$ due to the assumption $d \leq 3$. As is well known, $-A_0$ has compact resolvent; moreover, it is the infinitesimal generator of an analytic semigroup [8].

The maps $b_0(\cdot)$ and $k_0(t,\cdot)$ $(t \geq 0)$ are assumed to be continuous and nonnegative on $\bar{\Omega}$, thus corresponding multiplication operators b and $k(t)$ on the space X are defined in an obvious way; in addition, $k_0(\cdot,\cdot)$ will be assumed to be measurable in $(t,x) \in (0,+\infty) \times \Omega$. Then we can rewrite problem (1) in the following abstract form:

$$(3) \quad \begin{cases} \dfrac{du}{dt} = -A_0 u - u \displaystyle\int_{-\infty}^{t} ds\, k(t-s)\, u(s) - bu^2 & (t > 0) \\ u = u_0 & (t \leq 0). \end{cases}$$

Concerning the solutions of problem (3), the following theorem holds.

THEOREM 1. *Let* $k(\cdot) \in L^1(0, +\infty; \mathscr{L}(X))$. *For any continuous, bounded nonnegative* $u_0 : (-\infty, 0] \to X$ *there exists a unique, nonnegative, global mild solution to problem* (3).

The continuous embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow X$ has been taken into account in the above statement; the Hölder continuity of $(-A_0 u)(\cdot)$ and $(du/dt)(\cdot)$ can be proved under additional regularity assumptions (for this purpose, no requirements on the space dimension d are needed [4]).

An equilibrium solution of the integro-differential equation in (3) is by definition a solution of the elliptic problem:

$$(4) \qquad -A_0 u - \left(\int_0^{+\infty} dt\, k(t) + b \right) u^2 = 0 .$$

If the principal eigenvalue λ of $-A_0$ is positive, a unique nontrivial solution $\hat{u} \in X$ of (4) exists [3, 7]; we shall limit ourselves to investigate the stability of such equilibrium (the stability properties of the trivial equilibrium being elementary).

An equilibrium solution $u \in X$ is said to be X-asymptotically stable with respect to (3) if it is both X-stable (namely, for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\sup_{t \leq 0} |u_0(t) - u|_X < \delta_\varepsilon$ implies $\sup_{t > 0} |u(t) - u|_X < \varepsilon$) and X-attractive (i.e., there exists $\eta > 0$ such that $\sup_{t \leq 0} |u_0(t) - u|_X < \eta$ implies $|u(t) - u|_X \to 0$ as $t \to +\infty$).

We can now state the main result, concerning the X-asymptotical stability of the nontrivial equilibrium solution of (4) (in the case $\lambda > 0$).

THEOREM 2. *Let* $\lambda > 0$; *assume moreover*:

$$k_1) \qquad \int_0^{+\infty} dt\, |k_0(t,\cdot)|_X < +\infty ,$$

$$\int_{-\infty}^{0} ds\, |k_0(t-s,\cdot) - k_0(t'-s,\cdot)|_X \leq \text{const.}\, |t-t'|^{\beta}$$

with $\beta \in (0, 1]$;

k_2) *there exists* $\alpha > (d-2)/2$ *such that*:

$$\max\{|(tk)^*(\zeta,\cdot)|_X, |(t^2 k)^*(\zeta,\cdot)|_X\} \le \text{const.}(1+|\zeta|)^{-\alpha}$$

for any $\zeta \in C$, $\mathrm{Re}\,\zeta \ge 0$;

H) *for any* $\zeta \in C$, $\mathrm{Re}\,\zeta \ge 0$, *there exists* $[\zeta I + A + \hat{u}k^*(\zeta)]^{-1} \in \mathscr{L}(X)$, *where* $A := A_0 + 2\hat{u}b + \hat{u}\int_0^{+\infty} dt\, k(t)$.

Then the nontrivial equilibrium $\hat{u}$ *of the Volterra equation in* (3) *is* X-*asymptotically stable.*

3. Proofs

The proof of Theorem 1 is easily sketched. The local existence result follows by [5, Theorem 2]; as for the uniqueness statement, it is a standard consequence of Gronwall's inequality, due to the local Lipschitz continuity of the nonlinear terms in the right hand-side. The nonnegativity of the solution, thus its existence in the large follow from the maximum principle for integral equations [11].

Let us notice the following lemma.

Lemma 1. *Let the above assumptions on* Ω *and* $-A_0$ *be satisfied. Then the following hold*:

A_1) $-A$ *is the infinitesimal generator of an analytic semigroup* $T(\cdot)$ *on* X;

A_2) *the operator* $(\zeta + A)^{-1}$ *is compact for any* $\zeta \in C$ *belonging to the resolvent set* $\rho(-A)$; *moreover, there exists a sequence* $\{P_n\} \subset \mathscr{L}(X)$ *of projections such that* ($n \in N$):

(i) $\dim P_n < \infty$, $P_n P_{n+1} = P_{n+1} P_n = P_n$;

(ii) $P_n A \subseteq A P_n$;

(iii) *there exist* $\vartheta_0 \in [0, 1)$ *and* $\zeta_0 \in \rho(-A)$ *such that* $(I - P_n)\cdot(\zeta_0 + A)^{-\vartheta_0}$ *converges in the strong sense as n diverges*;

(iv) *there exists* $\bar{n} \in N$ *such that, for any* $n \ge \bar{n}$, *the type* ω_n *of the analytic semigroup* $(I - P_n)T(\cdot)$ *is strictly negative.*

Proof. A_1) follows from [8] by standard results; the compactness statement in A_2) is a well known consequence of the Rellich theorem. In order to introduce the projections P_n, it is convenient to think of A as of an operator in $L^2(\Omega)$, with $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$. Due to well known representation theorems, we get $-Av = \sum_{i=1}^{\infty} \lambda_i (v, \varphi_i)\varphi_i$ for any $v \in L^2(\Omega)$ ($(\cdot, \cdot)$ denoting the usual L^2-scalar product), where the eigenvalues λ_i go to $-\infty$ as $i \to \infty$, and the eigenfunctions φ_i of $-A$ are continuous on Ω by classical regularity results. Now we claim that the family $\{P_n\} \in \mathcal{L}(X)$, $v \to P_n v := \sum_{i=1}^{n} (v, \varphi_i)\varphi_i$ satisfies the requirements A_2), (i)-(iv); only the statement A_2), (iii) deserves further investigation. Observe that, for any $\vartheta_0 > d/4$, the space $H^{2\vartheta_0} := = \left\{ v \in L^2(\Omega) \;\middle|\; \sum_{i=1}^{\infty} |\lambda_i|^{2\vartheta_0} |(v, \varphi_i)|^2 < \infty \right\}$ is continuously embedded into X by Sobolev inequalities; moreover, it is easily seen that $P_n : H^{2\vartheta_0} \to H^{2\vartheta_0}$ and P_n strongly converges to the identity in the $H^{2\vartheta_0}$-norm as n diverges. Then the conclusion follows from the continuous embedding $H^{2\vartheta_0} \hookrightarrow X$, due to the assumption $d \leq 3$.

Consider the following linear integro-differential equation in X:

$$\frac{du}{dt} = -Au - \hat{u} \int_0^t ds\, k(t-s)\, u(s) \qquad (t \geq 0) \tag{5}$$

($\hat{u}$ denoting the multiplication operator on X, $v(\cdot) \to (\hat{u}v)(\cdot) := \hat{u}(\cdot)\, v(\cdot)$). According to [6], there exists a unique fundamental solution $t \to \mathcal{S}(t)$ $(t \geq 0)$ such that $\mathcal{S}(t)w$ is the unique mild solution of (5) equal to w for $t = 0$.

We can now prove the main result.

Proof of Theorem 2. It suffices to prove that $\|\mathcal{S}(t)\| \leq c/(1 + t^2)$ $(c > 0; t \geq 0)$; then $\mathcal{S}(\cdot) \in L^1(0, +\infty; \mathcal{L}(X))$ and the result follows by a standard linearization argument [1, 9]. Due to the assumption k_2), the above inequality follows by the same arguments as in [6, Propositions 4,9].

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