Olusola Akinyele

Conditionally asymptotically invariant sets and perturbed systems

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **67** (1979), n.3-4, p. 214–222.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_67_3-4_214_0>

Equazioni differenziali ordinarie. — *Conditionally asymptotically invariant sets and perturbed systems.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore considera un insieme condizionalmente assintoticamente invariante rispetto a un sistema S di equazioni differenziali. Supposto inoltre l'insieme uniformemente assintoticamente stabile ne ricava alcune proprietà. Studia anche il caso in cui S venga perturbato.

§ 1. INTRODUCTION

In [2, 3, 4], the concept of conditionally invariant sets (CI) was introduced and their stability properties investigated. This led to the concept of conditionally asymptotically invariant sets (CAI) which was introduced in [5]. The stability behaviour of such sets thus gave rise to much weaker notions of stability; namely, extreme uniform stability and extreme uniform asymptotic stability of a set relative to another set, for which the construction of smooth Lyapunov functions becomes possible [5, § 3]. However, the question as to whether Massera's type of converse theorem when a CAI set relative to another set is uniformly asymptotically stable can be obtained was not investigated in [5] and the author is not aware of any such investigations.

In this paper, we therefore present the Massera's type converse theorem when a CAI set relative another set is uniformly asymptotically stable. In § 3, we employ our result to investigate the effects of constantly acting perturbations when a CAI set relative to another set is uniformly asymptotically stable with respect to the unperturbed system.

§ 2. MAIN RESULTS

We consider the differential system

$$\frac{dx}{dt} = f(t, x) \qquad x(t_0) = x_0 \tag{1}$$

where $f \in C(R^+ \times S(B, \rho), R^n)$, $R^+ = [0, \infty)$ and R^n is the Euclidean space. Here $S(B, \rho) = \{x \in R^n : d(x, B) < \rho\}$ $d(x, B) = \inf_{y \in B} \|x - y\|$, $\|\cdot\|$ being any convenient norm, B is a subset of R^n and $C(R^+ \times S(B, \rho), R^n)$ is the class of continuous functions from $R^+ \times S(B, \rho)$ to R^n.

(*) Nella seduta del 21 aprile 1979.

Corresponding to the differential system (1) we shall consider the perturbed differential system

$$\frac{dy}{dt} = f(t, y) + R(t, y) \qquad y(t_0) = y_0 \tag{2}$$

where $R \in C(R^+ \times S(B, \rho). R^n)$. Denote by $\mathscr{L}$ the class of functions $\lambda \in C(R^+, R^+)$ such that $\lambda(t)$ is decreasing in t and $\lim_{t\to\infty} \lambda(t) = 0$; by $\mathscr{K}$ the class of functions $\eta \in C([0, \rho), R^+)$ with $\eta(0) = 0$ and $\eta(r)$ is increasing in r and by $\mathscr{D}$ the class of functions $b \in C(R^+ \times [0, \rho), R^+)$ such that $b(t, r)$ is decreasing in t for each r, increasing in r for each t and $\lim_{\substack{t\to\infty \\ t\to 0^+}} b(t, r) = 0$.

DEFINITION 2.1. [5] Let $A, B, \subset R^n$ with $A \subset B$. The set B is said to be conditionally asymptotically invariant (CAI) relative to the set A and the differential system (1) if there exists a function $\lambda \in \mathscr{L}$ such that $x_0 \in A$ implies

$$d(x(t, t_0, x_0), B) \leq \lambda(t_0) \qquad t \geq t_0.$$

where $x(t; t_0, x_0)$ is a solution of (1) such that $x(t_0; t_0, x_0) = x_0$

DEFINITION 2.2. [5] The CAI set B relative to the set A and the differential system (1) is said to be.

(i) Uniformly stable if there exists $\alpha \in \mathscr{K}$ and $\lambda \in \mathscr{L}$ such that

$$d(x(t, t_0, x_0), B) \leq \alpha(d(x_0, A)) + \lambda(t_0) \qquad t \geq t_0.$$

(ii) Uniformly asymptotically stable if there exist fuctions $\beta \in \mathscr{K}$, $\gamma \in \mathscr{L}$ and $H \in C(R^+ \times R^+, R^+)$ such that $\lim_{t\to\infty} H(t, t_0) = 0$ provided $t_0 \geq \tau > 0$ for some τ and

$$d(x(t, t_0, x_0), B) \leq \beta(d(x_0, A))\, \gamma(t - t_0) + H(t, t_0); \qquad t \geq t_0. \tag{3}$$

Remark 1. In definition 2.1. $B \neq A$ corresponds to the asymptotic self-invariance of A (ASI), $\lambda(t) \equiv 0$ corresponds to the conditional invariance of B with respect to A, while $B = A$ and $\lambda(t) \equiv 0$ yield the self invariance of A. It follows then that definition 2.2 contains as special cases the uniform stability and uniform asymptotic stability definitions of the conditional invariant set B with respect to A or the self invariant set A. If $B = A = \{0\}$ and $H(t, t_0) \equiv 0$ then (3) becomes $\| x(t, t_0, x_0) \| \leq \beta(\| x_0 \|)\, \gamma(t - t_0)$, $t \geq t_0$, which is the uniform asymptotic stability definition of the set $x = 0$ [cf. 4].

In the next theorem we give a construction of the Massera's type Lyapunov functions. We assume conditions which ensure the existence and uniqueness of solutions of (1).

THEOREM 2.3 (i) *Assume that the CAI set* B *relative to* A *and the differential system* (1) *is uniformly asymptotically stable. In addition let*

$$\|f(t, x_1) - f(t, x_2)\| \leq L(t)\|x_1 - x_2\| \tag{4}$$

for $(t, x_1), (t, x_2) \in R^+ \times S(B, \rho)$ *where* $L(t)$ *is continuous on* R^+ *and*

$$\left|\int_t^{t+\theta} L(s)\, ds\right| \leq M(\theta).$$

(ii) *For* $H \in C(R^+ \times R^+, R^+)$, *let* $H(t, t) \equiv 0$ *and the function* $H(t, t_0)$ *is partially differentiable with respect to* t_0 *and*

$$\sup_{\sigma \geq 0} \left\{ -\frac{\partial H}{\partial t_0}(t + \sigma, t) \right\} \leq \eta(t) \tag{5}$$

where $\eta \in C(R^+, R^+)$ *and* $\int_t^{t+1} \eta(s)\, ds \to 0$ *as* $t \to \infty$. *Moreover,* $H(t, t_0) \leq q(t_0)$ *for some* $q \in \mathscr{L}$ *and* $\lim_{t \to \infty} \sup_{t_0 \geq T} H(t, t_0) = 0$ *for some* $T > 0$.

Then there exists a Lyapunov function $V(t, x)$ such that

(I) $V \in C(R^+ \times S(B, \rho), R^+)$ and for $(t, x_1), (t, x_2) \in R^+ \times S(B, \alpha)$

$$|V(t, x_1) - V(t, x_2)| \leq M(\alpha)\|x_1 - x_2\|$$

(II) there exist $a, b, \in \mathscr{K}$ such that

$$a(d(x, B)) \leq V(t, x) \leq b(d(x, A))$$

and (III)

$$D^+ V(t, x) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x + h \cdot f(t, x)) - V(t, x)]$$

$$\leq -C(d(x, B)) + k(t)$$

where $C \in \mathscr{K}$ and $k \in C(R^+, R^+)$ with $\int_t^{t+1} k(s)\, ds \to 0$ as $t \to \infty$.

Proof. Let $G(r)$ be a function defined and having the properties as contained in Theorem 3.6.9 of [4]. Using G, define a function $V(t, x)$ as follows:

$$V(t, x) = \sup_{\sigma \geq 0} G[d(x(t + \sigma, t, x), B) - H(t + \sigma, t)]\left(\frac{1 + \alpha\sigma}{1 + \sigma}\right) \tag{6}$$

where $x(t, t_0, x_0)$ is any solution of (1) with $x(t_0, t_0, x_0) = x_0$, and α is chosen such that $\alpha \geq \max\{1 + \varepsilon, G(\varepsilon(d(x, A)))\}$, $\varepsilon > 0$

(7) $a(d(x, B) \leq V(t, x)$ where $a \in \mathscr{K}$, if $\sigma = 0$ and we set $a(u) = G(u)$.

Now for $\sigma \geq 0$ and by hypothesis,

$$d(x(t+\sigma, t, x), B) \leq \beta(d(x, A))\gamma(\sigma) + H(t+\sigma, t)$$

where $\gamma \in \mathscr{L}$, $H \in C(R^+ \times R^+, R^+)$, $H(t, t) = 0$, $H(t, t_0) \leq q(t_0)$ for $q \in \mathscr{L}$ and $\lim_{t\to\infty} \sup_{t_0 \geq T} H(t, t_0) = 0$ for some $T > 0$. Moreover, $\dfrac{1 + \alpha\sigma}{1 + \sigma} < \alpha$ and if $\varepsilon(\delta)$ is the inverse function of $\delta(\varepsilon)$, then by the assumption (i), $d(x(t+\sigma, t, x,) B) - H(t+\sigma, t) \leq \beta(d(x, A))\gamma(\sigma)$ and an argument similar to that of Theorem 3.4.11 of [4] yields

$$d(x(t+\sigma, t, x), B) - H(t+\sigma, t) \leq \varepsilon(d(x, A)).$$

$$\therefore \sup_{\sigma \geq 0} G[d(x(t+\sigma, t, x), B) - H(t+\sigma, t)] \frac{1 + \alpha\sigma}{1 + \sigma} \leq \alpha G(\varepsilon(d(x, A))).$$

If we set $b(u) = \alpha G(\varepsilon(u))$: we obtain,

$$V(t, x) \leq b(d(x, A)), \tag{8}$$

so that (7) and (8) imply (II). Let $\tau = \dfrac{d(x, B)}{\alpha\beta(d(x, A))}$; then if $\sigma \geq T(\tau)$, since $\gamma \in \mathscr{L}$, $\gamma(\sigma) \leq \tau$ and so

$$G[d(x(t+\sigma, t, x), B) - H(t+\sigma, t)] \frac{1 + \alpha\sigma}{1 + \sigma} < \alpha G[\beta(d(x, A)) \cdot \tau] \leq V(t, x).$$

It follows therefore that

$$V(t, x) = \sup_{0 \leq \sigma \leq T(\tau)} G[d(x(t, +\sigma, t, x) B) - H(t+\sigma, t)] \frac{1 + \alpha\sigma}{1 + \sigma}.$$

By continuity of V, we can find a σ_1 such that

$$V(t, x) = G[d(x(t+\sigma_1, t, x), B) - H(t+\sigma_1, t) \frac{1 + \alpha\sigma_1}{1 + \sigma_1},$$

and if we let $x = x(t, t_0, x_0)$ and $x^* = x(t+h, t, x)$, then by the uniqueness of solutions,

$$V(t+h, x^*) = G[d(x(t+h+\sigma^*, t+h, x^*), B) -$$

$$- H(t+h+\sigma^*, t+h)] \frac{1 + \alpha\sigma^*}{1 + \sigma^*} =$$

$$= G[d(x(t+h+\sigma^*, t, x), B) - H(t+h+\sigma^*, t)] \frac{1 + \alpha\sigma^*}{1 + \sigma^*}.$$

Set $\sigma^* + h = \sigma$, then it can be easily shown (cf, 4] that

$$\frac{1+\alpha\sigma^*}{1+\sigma^*} = \frac{1+\alpha\sigma}{1+\sigma}\left[1 - \frac{(\alpha-1)h}{(1+\sigma^*)(1+\alpha\sigma)}\right].$$

$$\therefore\ V(t+h, x^*) = G\,[d(x(t+\sigma, t, x), B) -$$
$$- H(t+\sigma, t+h)]\left(\frac{1+\alpha\sigma}{1+\sigma}\right)\left[1 - \frac{(\alpha-1)h}{(1+\sigma^*)(1+\alpha\sigma)}\right].$$

Now applying then mean value theorem to $H(t, t_0)$ in the second variable t_0;

$$H(t+\sigma, t+h) = H(t+\sigma, t) + h\cdot\frac{\partial H}{\partial t_0}(t+\sigma, t+\theta h),\ 0 < \theta < 1 \quad \text{and}$$

$$V(t+h, x^*) \le G\,[d(x(t+\sigma, t, x), B) - H(t+\sigma, t) +$$
$$+ h\eta(t+\theta h)]\left(\frac{1+\alpha\sigma}{1+\sigma}\right)\times\left(1 - \frac{(\alpha-1)h}{(1+\sigma^*)(1+\alpha\sigma)}\right).$$

For sufficiently small h,

$$G(d(x(t+\sigma, t, x), B) - H(t+\sigma, t) + h\eta(t+\theta h)) =$$
$$= G(d(x(t+\sigma, t, x), B) - H(t+\sigma, t)) + h\eta(t+\theta h)\,G'\,[d(x, B) -$$
$$- H(t+\sigma, t) + \varphi h\eta(t+\theta h)],\ 0 < \varphi < 1.$$

Hence,

$$V(t+h, x^*) \le V(t, x)\left\{1 - \frac{(\alpha-1)h}{(1+\sigma^*)(1+\alpha\sigma)}\right\} +$$
$$+ h\eta(t+\theta h)\,G'(r)\left(\frac{1+\alpha\sigma}{1+\sigma}\right)\times\left\{1 - \frac{(\alpha-1)h}{(1+\sigma^*)(1+\alpha\sigma)}\right\},$$

where $r = d(x(t+\sigma, t, x), B) - H(t+\sigma, t) + \varphi h\eta(t+\theta h)$. Consequently

$$\frac{V(t+h, x^*) - V(t, x)}{h} \le -\frac{(\alpha-1)V(t, x)}{(1+\sigma^*)(1+\alpha\sigma)} +$$
$$+ \eta(t+\theta h)\,\alpha\cdot G'(r)\left(1 - \frac{(\alpha-1)h}{(1+\sigma^*)(1+\alpha\sigma)}\right).$$

Since $0 \le \sigma^* < T(\tau)$, then $0 < \sigma \le T(\tau) + h$, hence by continuity of η and G' on R^+,

$$D^+V(t, x) \le \frac{-(\alpha-1)\,G(d(x, B))}{(1+T(\tau))(1+\alpha T(\tau))} + \eta(t)\,\alpha\,G'(d(x, B) - H(t+\sigma, t)).$$

Clearly $G'(d(x, B) - H(t+\sigma, t)) \leq \alpha G'(\varepsilon(d(x, A)))$ because of the monotonic increasing property of G', hence

$$D^+ V(t, x) \leq -\frac{(\alpha - 1) G(d(x, B))}{(1 + T(\tau)(1 + \alpha T(\tau))} + \eta(t)\, \alpha^2 G'(\varepsilon(d(x, A))).$$

So if we set $-C(d(x, B)) = -\dfrac{(\alpha - 1) G(d(x, B))}{(1 + T(\tau))(1 + \alpha T(\tau))}$ and $k(t) = \alpha^3 \eta(t)$,

$$D^+ V(t, x) \leq -C(d(x, B)) + k(t)$$

where $C \in \mathscr{K}$, $k \in C(R^+, R^+)$ and $\int_t^{t+1} k(s)\, ds \to 0$ as $t \to \infty$, since $\lim\limits_{h \to 0^+}$ $d(x(t+h, t, x), B) = d(x(t, t, x) B)$ and $T(\tau)$ is a decreasing function.

Let x_1, x_2 be such that solutions $x(t, t_0, x_1)$ and $x(t, t_0, x_2) \in S(B, \alpha)$. Let r_1 and r_2 be defined as follows;

$$r_1 = d(x(t+\sigma_1, t, x_1), B) - H(t+\sigma_1, t) \text{ and } r_2 =$$
$$= d(x(t+\sigma_1, t, x_2), B) - H(t+\sigma_1, t).$$

Then by assumption (i) and Corollary 2.7.1 of [4]

$$\| x(t+\sigma_1, t, x_1) - x(t+\sigma_1, t, x_2) \| \leq \exp MT \frac{\| x_1 \|}{\alpha} \| x_1 - x_2 \|.$$

With these modifications the remaining part of the proof can be worked out in a similar way to that of Theorem 3.6.9 of [4].

Remark 2. If $\lambda(t) \equiv 0$ in definition 2.1, then out theorem reduces to the converse theorem for the conditionally invariant set B with respect to A and the differential system (1). If $B = A$ then we have the converse theorem for the ASI set A, and $B = A$ with $\lambda(t) \equiv 0$ yield the same theorem for self-invariant set A. On the otherhand if $B = A = \{0\}$ and $H(t, t_0) \equiv 0$ our result reduces to Theorem 3.6.9 of [4] for the set $x = 0$. Our result also settles the question asked by Ladde and Lakshmikantham in [5, Remark 2.3] concerning Massera's type of converse theorem when a CAI set or CI set is uniformly asymptotically stable. We also note that the notion of uniform asymptotic stability as contained in this paper is weaker than the extreme uniform stability of [5]. It can also be shown that there exists $C^* \in \mathscr{K}$ and $V(t, x) \in C(R^+ \times \times S(B, \rho), R^+)$ such that $D^+ V(t, x) \leq -C^*(V(t, x)) + k(t)$, using the last result.

§ 3. PERTURBATION RESULTS

Here we shall utilize the result of the section 2 to investigate the preservation of stability behaviour of CAI set B relative to A and the differential system (1) under constantly acting perturbations.

Along with the given differential system (1) we consider the scalar differential equation

$$\frac{du}{dt} = g(t, u) \qquad u(t_0) = u_0 > 0 \tag{9}$$

where $g \in C(R^+ \times R^+, R)$. We assume that the set $u = 0$ is asymptotically self-invariant (ASI) relative to (9).

THEOREM 3.1. *Assume that the* CAI *set* B *with respect to* A *and the system* (1) *is uniformly asymptotically stable and that all the hypothesis of Theorem 2.3 hold. Let* $R(t, x)$ *of the system* (2) *satisfy the inequality*

$$\| R(t, x) \| \leq \lambda_\alpha(t) \tag{10}$$

whenever $x \in \overline{S(B, \alpha)} \sim A$ *and such that*

$$\int_{t_0}^{t} \lambda_\alpha(s)\, ds \leq q_0(t_0) + q_1(t_0)(t - t_0) \qquad \text{for some } \; q_0, q_1 \in \mathscr{L}. \tag{11}$$

Then the set B is CAI relative to the set A and the perturbed system (2) and it is uniformly asymptotically stable.

Proof. For $(t, x) \in R^+ \times S(B, \alpha)$ it is routine to show that

$$D^+ V(t, x)_{(2)} \leq D^+ V(t, x)_{(1)} + M(\alpha) \| R(t, x) \|$$

where $V(t, x)$ is the Lyapunov function of Theorem 2.3 and $D^+ V(t, x)_{(2)}$ denotes its Dini derivative with respect to the system (2). By the property (III) of $V(t, x)$ in Theorem 2.3 and condition (10) on R,

$$D^+ V(t, x)_{(2)} \leq -C(d(x, B)) + k(t) + M(\alpha) \lambda_\alpha(t)$$

with $C \in \mathscr{K}$. Now $\int_{t}^{t+1} (k(s) + M(\alpha) \lambda_\alpha(s))\, ds \to 0$ as $t \to \infty$, hence there exists $\beta \in \mathscr{L}$ such that $\int_{t_0}^{t} (k(s) + M(\alpha) \lambda_\alpha(s))\, ds \leq \beta(t_0)$ for $t \geq t_0$ and it follows that

$$D^+ V(t, x)_{(2)} \leq k(t) + M(\alpha) \lambda_\alpha(t)$$

implies that the solution $u(t, t_0, u_0)$ of (9) with $g(t, u) = k(t) + M(\alpha)\lambda_\alpha(t)$ satisfies the inequality

$$u(t, t_0, u_0) \leq u_0 + p(t_0) \qquad t \geq t_0, \quad p \in \mathscr{L}. \tag{12}$$

Clearly (12) implies that the set $u = 0$ of (9) is ASI and uniformly stable. An application of Theorem 2.1 of [5] then implies that the set B is CAI with respect to the set A and the perturbed system (2) and it is also uniformly stable.

We now show asymptotic stability of the CAI set B. It suffices to show that for each $t_0 \in R^+$, there exists $t^* \in [t_0, t_0 + T(\varepsilon)]$ for some $T(\varepsilon)$ such that

$$d(x(t^*, t_0, x_0), B) \leq \beta(d(x_0, A))\gamma(t - t_0), \qquad t \geq t_0 \tag{13}$$

where $T(\varepsilon)$ is a positive number depending on $\varepsilon > 0$.

Set $\delta = \delta(\rho)$ and let $0 < \varepsilon < \rho$ and $t_0 \in R^+$. By Theorem 2.3 let $b \in \mathscr{K}$ satisfy property (II) and choose $T(\varepsilon) = \dfrac{4\,b(\rho)}{C(\delta(\varepsilon))}$. We claim that for the choice of T inequality (13) holds. Suppose not; then

$$d(x(t, t_0, x_0), B) > \beta(d(x_0, A))\gamma(t - t_0) + H(t, t_0) \tag{14}$$

for all $t \in [t_0, t_0 + T(\varepsilon)]$. By Theorem 2.3,

$$D^+ V(t, x)_{(2)} \leq -C(d(x, B) + k(t) + M(\alpha)\lambda_\alpha(t) \tag{15}$$

for $t \in [t_0, t_0 + T(\varepsilon)]$ so that if we integrate (15) from t_0 to $t_0 + T(\varepsilon)$, we obtain

$$V(t_0 + T(\varepsilon), x(t_0 + T(\varepsilon), t_0, x_0)) \leq V(t_0, x_0) -$$

$$-\int_{t_0}^{t_0+T(\varepsilon)} C(\beta(d(x_0, A))\gamma(s - t_0) + H(s, t_0))\,ds + \int_{t_0}^{t_0+T(\varepsilon)} k(s)\,ds + \int_{t_0}^{t+_0T(\varepsilon)} M(\alpha)\,\tau_\alpha(s)\,ds$$

Given $\delta(\varepsilon)$, then $\beta(d(x_0, A))\gamma(T(\varepsilon)) + H(t_0 + T(\varepsilon), t_0) \geq \delta(\varepsilon) > 0$, and in view of (11)

$$0 < a(\delta(\varepsilon)) \leq b(\rho) - C(\delta(\varepsilon))T(\varepsilon) + \int_{t_0}^{t_0+T(\varepsilon)} k(s)\,ds + M(\alpha)q_0(t_0) +$$

$$+ M(\alpha)q_1(t_0)T(\varepsilon).$$

Since $\displaystyle\int_{t_0}^{t+1} k(s)\,ds \to 0$ as $t \to \infty$, we can choose t_0 sufficiently large such that

$$\int_{t_0}^{t_0+T(\varepsilon)} k(s)\,ds \leq \frac{b(\rho)}{2}, \quad q_0(t_0) \leq \frac{b(\rho)}{2M(\alpha)} \quad \text{and} \quad q_1(t_0) \leq \frac{C(\delta(\varepsilon))}{2M(\alpha)}.$$

In that case, $0 < a(\delta(\varepsilon)) \leq b(\rho) - C(\delta^{*}(\varepsilon)) \frac{4\, b(\rho)}{C(\delta(\varepsilon))} + \frac{b(\rho)}{2} + \frac{b(\rho)}{2} + \frac{C(\delta(\varepsilon))\, 4b(\rho)}{2\, C(\delta(\varepsilon))} = 0$ which is a contradiction and the proof is complete.

Remark 3. If $B = A = \{0\}$ and $H(t, t_0) \equiv 0$, then our result reduces to Theorem 3.3 of [1], since the uniform asymptotic stability of the ASI set $x = 0$ is equivalent to the eventual uniform asymptotic stability of the trivial solution of the system (1). Our theorem also gives the perturbation results for the CI set B with respect to A corresponding to the special case $\lambda(t) \equiv 0$ in definition 2.1. In view of remarks 1 and 2 and [1; § 3 Remark] our result includes a wide range of perturbation results, [cf, 1, 6, 7].

REFERENCES

[1] O. AKINYELE (1977) – *On the eventual stability of perturbed non-liner systems*, «Ann. Mat. Pura ed Appl.» (4), *112*, 261–271.

[2] S. G. DEO and G. S. LADDE (1974) – *Conditionally invariant sets and partial stability*, «Indian J. Of Pure and Appl. Maths.», (8), *5*, 755–757.

[3] A. A. KAYANDE and V. LAKSHMIKANTHAM (1966) – *Conditionally invariant sets and vector Lyapunuv function*, «J. Math. Anal and App.», *13*, 337–347. Vol. 1, Academic Press, New York.

[4] V. LAKSHMIKANTHAM and S. LEELA (1969) – *Differential and integral inequalities*. Vol. 1, Academic Press, New York.

[5] G. S. LADDE, V. LAKSHMINKANTHAM and S. LEELA (1972) – *Conditionally Asymptotically invariant sets and perturbed systems*, «Ann. Mat. Pura ed Appl.» (4), *94*, 33–40.

[6] S. LEELA (1972) – *Asymptotically self invariant sets and perturbed systems*, «Ann. Mat. Pura ed Appl.», (4), *92*, 85–93.

[7] SHNI–NEW CHOW and J. A. YORKE (1974) – *Lyapunov theory and perturbation of stable and asymptotically stable systems*, «J. of Differential Equations», *15*, 308–321.