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The equations $y' = Sy + T$ and $By' = Sy + T$ with B, S, T distributions

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Analisi funzionale. — *The equations* $y' = Sy + T$ *and* $By' = Sy + T$ *with* B, S, T *distributions.* Nota (*) di PIERO PLAZZI (**), presentata dal Corrisp. G. CIMMINO.

RIASSUNTO. — Fondandosi su una definizione di prodotto proposta da E. E. Rosinger si studiano le equazioni ordinarie $y' = Sy + T$, $By' = Sy + T$ con B, S, T distribuzioni.

INTRODUCTION

In this paper we shall use the general framework of [1], which is necessary for the understanding of our work; in this monography the author constructs linear algebras which contain vector subspaces canonically isomorphic to $\mathscr{D}'(R^n)$ so that a commutative and associative product may be defined between distributions.

The main aim of this construction seems to be the study of nonlinear partial differential equations, but the method allows us to consider also linear differential equations with irregular coefficients, while generally in the framework of usual distribution theory only C^∞ coefficients are allowed.

In this paper we consider two simple Cauchy problems relative to ordinary first-order differential equations, namely $y' = Sy + T$, $By' = Sy + T$: they involve irregular products with distributions B, S, T.

The properties required to the algebras play an essential rôle in the proofs of existence and uniqueness results: for the former problem such a result is presented in Theorem 4, while an existence result for the latter one is proved in Theorem 5.

NOTATIONS

We shall use systematically Rosinger's notations: we refer to [1], particularly chapter 1, throughout the whole paper; here are only a few particular notations.

We denote by N the set of positive integers, and $N_0 = N \cup \{0\}$, so in particular we put $W = [C^\infty(R^n)]^N = W(R^n)$: if $\boldsymbol{s} \in W$ we write $\boldsymbol{s} = (s_\nu)_{\nu \in N}$ and $\boldsymbol{s}(x_0) = (s_\nu(x_0))_{\nu \in N}$ $\forall x_0 \in R^n$; if $\alpha \in N_0^n$ (a multi-index) we write $D^\alpha \boldsymbol{s} = (D^\alpha s_\nu)_{\nu \in N}$ and so on. Moreover, we shall write $A \leq B$ if A is a (linear) subalgebra of B.

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Finally, if $z \in \mathbf{C}$, $z \neq 0$, Arg z denotes the principal argument of z, so Arg $z \in]-\pi, \pi]$; $[x]$ is the entire part of $x \in \mathrm{R}$.

First, we define certain classes of subalgebras in W, and prove some properties of them, described in Theorem 1.

DEFINITION 1. *An algebra* $\mathrm{A} \leq \mathrm{W}$ *is* $\mathscr{E}$*–closed iff* $\boldsymbol{s}_k \in \mathrm{A}$ $\forall k \in \mathrm{N}$, $\boldsymbol{s}_k = (s_{k\nu})_{\nu \in \mathrm{N}}$, $\forall \nu \in \mathrm{N} \, s_{k\nu} \xrightarrow[k\to\infty]{} s_\nu$ *uniformly on compact subsets of* R^n *with all derivatives, imply* $\boldsymbol{s} = (s_\nu)_{\nu \in \mathrm{N}} \in \mathrm{A}$.

THEOREM 1. *Suppose* $\mathrm{A} \leq \mathrm{W}$ *is closed for affine changes of variables, closed for multiplication by the coordinates* (i.e. $(s_\nu)_{\nu\in\mathrm{N}} \in \mathrm{A} \Rightarrow (s_\nu \circ \mathrm{C})_{\nu\in\mathrm{N}}$ *and* $(\mathrm{P}_j s_\nu)_{\nu\in\mathrm{N}} \in \mathrm{A}$, *where* $\mathrm{C}(x) = \mathscr{A}x + b$, $\mathrm{P}_j(x) = x_j \, \forall x \in \mathrm{R}^n$, $\forall j \in \{1, \cdots, n\}$, *with* $\mathscr{A}$ *invertible* $n \times n$ *real matrix*, $b \in \mathrm{R}^n$) *and* $\mathscr{E}$*–closed. Then*

i) $\boldsymbol{s} \in \mathrm{A} \Rightarrow \partial \boldsymbol{s}/\partial x_j \in \mathrm{A}$ $\forall j \in \{1, \cdots, n\}$ (A *is derivative invariant*: see [1] p. 14);

ii) *with* $a \in \mathrm{R}$ *fixed, put* $\mathrm{T}_p(f)(x) = \int_a^{x_p} f(x_1, \cdots, x_{p-1}, t, x_{p+1}, \cdots, x_n)\, \mathrm{d}t$ $(x \in \mathrm{R}^n, f \in \mathrm{C}(\mathrm{R}^n, \mathbf{C}), p \in \{1, \cdots, n\})$; *then* $(s_\nu)_{\nu\in\mathrm{N}} \in \mathrm{A} \Rightarrow (\mathrm{T}_p(s_\nu))_{\nu\in\mathrm{N}} \in \mathrm{A}$;

iii) $\boldsymbol{s} \in \mathrm{A}$, $\forall \nu \in \mathrm{N}\; s_\nu(\mathrm{R}^n) \subseteq \Omega = \mathring{\Omega} \subseteq \mathbf{C}$ *with* Ω *simply connected*, f *holomorphic in* $\Omega \Rightarrow f(\boldsymbol{s}) = (f \circ s_\nu)_{\nu\in\mathrm{N}} \in \mathrm{A}$.

Proof. i) Put $\Sigma_k(f)(x) = k(f(x + k^{-1} e_j) - f(x))$ with $f \in \mathrm{C}^\infty(\mathrm{R}^n)$, $k \in \mathrm{N}$, $x \in \mathrm{R}^n$ and $e_j = (0, \cdots, \underset{j}{1}, \cdots, 0)$; we have $\dfrac{\partial s_\nu}{\partial x_j}(x) = \lim\limits_{k\to\infty} \Sigma_k(s_\nu)(x)$. By hypothesis, $(\Sigma_k(s_\nu))_{\nu\in\mathrm{N}} \in \mathrm{A}$ $\forall k \in \mathrm{N}$; if $\alpha \in \mathrm{N}_0^n$, $\mathrm{D}^\alpha \dfrac{\partial s_\nu}{\partial x_j}(x) = \lim\limits_{k\to\infty} \Sigma_k(\mathrm{D}^\alpha s_\nu)(x)$, so we have only to prove $\Sigma_k(f) \xrightarrow[k\to\infty]{} \dfrac{\partial f}{\partial x_j}$ $\forall f \in \mathrm{C}^\infty(\mathrm{R}^n)$ uniformly on compact sets, which is trivial.

ii) Up to an affine change of variables, we may assume $p = n$. Consider now for $f \in \mathrm{C}^\infty(\mathrm{R}^n)$

$$\sigma_k(f)(t, x) = \sum_{j=1}^{k} \frac{x-a}{k} f\left(t, a + \frac{j}{k}(x-a)\right), \quad t \in \mathrm{R}^{n-1}, \; x \in \mathrm{R};$$

since $\boldsymbol{s} \in \mathrm{A} \Rightarrow (\sigma_k(s_\nu))_{\nu\in\mathrm{N}} \in \mathrm{A}$, we must only prove

$$\sigma_k(f)(t, x) \xrightarrow[k\to\infty]{} \int_a^x f(t, u)\, \mathrm{d}u$$

with all derivatives, uniformly on compact sets, $\forall f \in \mathrm{C}^\infty(\mathrm{R}^n)$.

First put $a_j = a_{j,k}(x) = a + \frac{j}{k}(x - a)$; then

$$\sigma_k(f)(t, x) - \int_a^x f(t, u)\, du = \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} [f(t, a_j) - f(t, u)]\, du ,$$

so

$$\left| \sigma_k(f)(t, x) - \int_a^x f(t, u)\, du \right| \leq |x - a| \sup_{|z-w| \leq (1/k)|x-a|} |f(t, z) - f(t, w)| \xrightarrow[k\to\infty]{} 0$$

uniformly if $|x - a| \leq$ const. and $t \in K$, a compact subset of R^n. Now observe

$$\sigma_k(D^\alpha f) = D^\alpha \sigma_k(f) \quad \text{if} \quad \alpha_n = 0,$$

so we have only to prove

$$D_n^r \sigma_k(f) \xrightarrow[k\to\infty]{} D_n^{r-1} f \qquad \forall r \in N$$

uniformly on compact sets. Now compute

$$D_n^r \sigma_k(f)(t, x) =$$

$$= r \sum_{j=1}^{k} \frac{1}{k} D_n^{r-1} f(t, a_j) \left(\frac{j}{k}\right)^{r-1} + \sum_{j=1}^{k} \frac{x - a}{k} (D_n^r f)(t, a_j) \left(\frac{j}{k}\right)^r$$

since $\frac{j}{k} = \frac{a_j - a}{x - a}$ we have for $x \neq a$

$$(1) \qquad \sum_{j=1}^{k} \frac{1}{k} D_n^{r-1} f(t, a_j) \left(\frac{j}{k}\right)^{r-1} \xrightarrow[k\to\infty]{} (x - a)^{-r} \int_a^x (D_n^{r-1} f)(t, u)(u - a)^{r-1} du =$$
$$= \sigma_1(t, x)$$

$$(2) \qquad \sum_{j=1}^{k} \frac{x - a}{k} (D_n^r f)(t, a_j) \left(\frac{j}{k}\right)^r \xrightarrow[k\to\infty]{} (x - a)^{-r} \int_a^x (D_n^r f)(t, u)(u - a)^r du =$$
$$= \sigma_2(t, x);$$

σ_1, σ_2 are continuous functions in R^n and $(r\sigma_1 + \sigma_2)(t, x) = D_n^{r-1} f(t, x)$, so we have only to prove that the limits (1) and (2) are uniform on compact subsets of R^n. Call Σ_1, Σ_2 the left-hand sides in (1), (2); then, if $a < x$

$$\Sigma_1 - \sigma_1 =$$

$$= \sum_{j=1}^{k} \left(\int_{a_{j-1}}^{a_j} [(D_n^{r-1} f)(t, a_j)(a_j - a)^{r-1} - (D_n^{r-1} f)(t, u)(u - a)^{r-1}]\, du \right) (x - a)^{-r} =$$

$$= (x-a)^{-r} \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} [(D_n^{r-1} f)(t, a_j)] [(a_j - a)^{r-1} - (u-a)^{r-1})] \, du +$$

$$+ (x-a)^{-r} \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} (u-a)^{r-1} [(D_n^{r-1} f)(t, a_j) - (D_n^{r-1} f)(t, u)] \, du = I_1 + I_2 .$$

We note that $k^{-r} \sum_{q=1}^{k} q^{r-1} \xrightarrow[k \to +\infty]{} r^{-1}$ $\forall r \in N$: this follows from the simple inequality

$$\int_1^{k+1} (x-1)^{r-1} \, dx \leq \int_1^{k+1} [x]^{r-1} dx \leq \int_1^{k+1} x^{r-1} \, dx \qquad \forall k \in N .$$

Now, if $\| t \| \leq$ const., $0 < x - a \leq$ const.

$$| I_1 | \leq C (x-a)^{-r} \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} [(a_j - a)^{r-1} - (u-a)^{r-1}] \, du =$$

$$= C \left[k^{-1} \sum_{j=1}^{k} \left(\frac{j}{k} \right)^{r-1} - r^{-1} \right] \xrightarrow[k \to +\infty]{} 0.$$

Since for $\| t \| \leq$ const., $0 < x - a \leq$ const. we have

$$| D_n^{r-1} f(t, a_j) - D_n^{r-1} f(t, u) | \leq C_1 | a_j - u | ,$$

we get

$$| I_2 | \leq C_1 (x-a)^{-r} \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} (u-a)^{r-1} (a_j - u) \, du =$$

$$= C_1 (x-a)^{-r} r^{-1} \sum_{j=1}^{k} \left[-(a_{j-1} - a)^r (a_j - a_{j-1}) + \int_{a_{j-1}}^{a_j} (u-a)^r \, du \right] =$$

$$= C_1 r^{-1} (x-a) \left[(r+1)^{-1} - k^{-1} \sum_{j=1}^{k} \left(\frac{j-1}{k} \right)^r \right] \xrightarrow[k \to +\infty]{} 0 .$$

For $x < a$ the proof interchanges a and x, since the preceding estimates depend only on $x - a$; the proof of (2) is similar but simpler.

iii) By Runge's theorem, $f(z) = \lim_{k \to \infty} P_k(z)$ with P_k polynomials, uniformly on compact subsets of Ω; since $(P_k \circ s_\nu)_{\nu \in N} \in A$ $\forall k \in N$ we get easily the desired result.

Let $A \leq W(R)$ be as in Theorem 1; moreover, suppose $\alpha \in R$, $\boldsymbol{\beta} \in C^N \cap A$, $\boldsymbol{\beta} = (\beta_\nu)_{\nu \in N}$; if $\boldsymbol{s}, \boldsymbol{t} \in A$ consider the 'Cauchy problem'

$$(\mathrm{P}) \quad \begin{cases} \boldsymbol{y}' = \boldsymbol{s}\boldsymbol{y} + \boldsymbol{t} \\ \boldsymbol{y}(\alpha) = \boldsymbol{\beta} \end{cases}$$

of finding $\boldsymbol{y} \in W(R)$, $\boldsymbol{y} = (y_\nu)_{\nu \in N}$ such that

$$(\mathrm{P}_\nu) \quad \begin{cases} y_\nu' = s_\nu y_\nu + t_\nu \\ y_\nu(\alpha) = \beta_\nu . \end{cases}$$

We may write the unique solution of (P_ν) as

$$y_\nu(x) = \left(\int_\alpha^x t_\nu(v) \exp\left(- \int_\alpha^v s_\nu(u) \, \mathrm{d}u \right) \mathrm{d}v + \beta_\nu \right) \exp\left(\int_\alpha^x s_\nu(u) \, \mathrm{d}u \right)$$

so that by Theorem 1 $\boldsymbol{y} \in A$.

In other words, we may state

THEOREM 2. *Suppose* $A \leq W(R)$ *be such that* a) A *is* $\mathscr{E}$*–closed;* b) A *is closed by multiplication by* x; c) A *is closed by affine changements of the variable;* d) $C^N \leq A$: *then* $\forall \boldsymbol{\beta} \in C^N$ *the 'Cauchy problem'* (P) *has its unique solution in* A.

From now on we fix a Q-regularization (V, S^0) (see [1] p. 15) with Q implying a, c, d in Theorem 2; then $\forall p \in N_0$ $A^Q(V(p), S^0)$ satisfies $a-d$ in Theorem 2 (b is obvious since $U \leq A^Q(V(p), S^0)$ in any case) and consider the algebras $A_p = A^Q(V(p), S^0)/I^Q(V(p), S^0)$: we write $[\boldsymbol{s}]_p$ for $\boldsymbol{s} + I^Q(V(p), S^0)$ $\boldsymbol{s} \in A^Q(V(p), S^0)$, and introduce the algebra homomorphisms $(r \in N)$ and linear mappings:

$$\gamma_{r,r-1} : A_r \to A_{r-1}, \; \gamma_{r,r-1}([\boldsymbol{s}]_r) = [\boldsymbol{s}]_{r-1} \qquad \forall \boldsymbol{s} \in A^Q(V(r), S^0)$$

$$D_{r,r-1} : A_r \to A_{r-1}, \; D_{r,r-1}([\boldsymbol{s}]_r) = [\boldsymbol{s}']_{r-1} \qquad \forall \boldsymbol{s} \in A^Q(V(r), S^0)$$

(see [1], pp. 18–19).

Now consider

$$(\mathrm{P}^0) \quad \begin{cases} y' = \boldsymbol{s}y + \boldsymbol{t} \\ y(\alpha) = \boldsymbol{\beta} \end{cases}$$

with $\boldsymbol{s}, \boldsymbol{t} \in S^0$, $\boldsymbol{\beta} \in C^N$, $\alpha \in R$ fixed.

DEFINITION 2. *We call* r*–solution of* (P^0), $r \in N$, *an element* $y \in A_r$ *such that*

$$(\mathrm{P}_r^0) \quad \begin{cases} D_{r,r-1}\, y = [\boldsymbol{s}]_{r-1}\, \gamma_{r,r-1}\, y + [\boldsymbol{t}]_{r-1} \\ \boldsymbol{z}(\alpha) - \boldsymbol{\beta} \in I^Q(V(r-1), S^0) \end{cases}$$

for a suitable $\boldsymbol{z} \in A^Q(V(r), S^0)$, $y = [\boldsymbol{z}]_r$.

It is now possible to prove

THEOREM 3. *If for* $k \in N_0$ *holds*

$$(S_k) \qquad (w_\nu)_{\nu \in N} \in I^Q(V(k), S^0) \Rightarrow (W_\nu)_{\nu \in N} \in I^Q(V(k), S^0)$$

with $W_\nu(x) = \int_a^x w_\nu(u)\,du \; (\nu \in N, x \in R)$, *then if* y_1, y_2 *are* $(k+1)$*-solutions of the Cauchy problem* (P^0), *we have* $\gamma_{k+1,k}\, y_1 = \gamma_{k+1,k}\, y_2$.

Proof. Let $y_i = [\boldsymbol{z}_i]_{k+1}$, $i = 1, 2$ with $\boldsymbol{z}_i \in A^Q(V(k+1), S^0)$ such that $\boldsymbol{z}_i(\alpha) - \beta \in I^Q(V(k), S^0)$: we shall prove $\boldsymbol{z}_1 - \boldsymbol{z}_2 \in I^Q(V(k), S^0)$. By (P^0_{k+1})

$$\boldsymbol{z}_i' + I^Q(V(k), S^0) = (\boldsymbol{s}\boldsymbol{z}_i + \boldsymbol{t}) + I^Q(V(k), S^0), \qquad i = 1, 2,$$

so that

$$(\boldsymbol{z}_1 - \boldsymbol{z}_2)' + I^Q(V(k), S^0) = \boldsymbol{s}(\boldsymbol{z}_1 - \boldsymbol{z}_2) + I^Q(V(k), S^0),$$

whence, with $\boldsymbol{w} = \boldsymbol{z}_1 - \boldsymbol{z}_2$

$$(3) \qquad \boldsymbol{w}' = \boldsymbol{s}\boldsymbol{w} + \boldsymbol{i}, \qquad \boldsymbol{i} \in I^Q(V(k), S^0);$$

moreover,

$$(4) \qquad \boldsymbol{w}(\alpha) = (\boldsymbol{z}_1(\alpha) - \beta) - (\boldsymbol{z}_2(\alpha) - \beta) \in I^Q(V(k), S^0).$$

Now from (3) and (4) we get

$$w_\nu(x) = \left(w_\nu(\alpha) + \int_\alpha^x i_\nu(v) \exp\left(-\int_\alpha^v s_\nu(u)\,du\right) dv\right) \exp\left(\int_\alpha^x s_\nu(u)\,du\right):$$

$$\left(\exp\left(\int_\alpha^{\bullet} s_\nu(u)\,du\right)\right)_{\nu \in N} \in A^Q(V(k), S^0)$$

and

$$\left(i_\nu \cdot \exp\left(-\int_\alpha^{\bullet} s_\nu(u)\,du\right)\right)_{\nu \in N} \in I^Q(V(k), S^0):$$

but by (S_k)

$$\left(\int_\alpha^{\bullet} i_\nu(v) \exp\left(-\int_\alpha^v s_\nu(u)\,du\right) dv\right)_{\nu \in N} \in I^Q(V(k), S^0),$$

whence $\boldsymbol{w} \in I^Q(V(k), S^0)$.

We introduce now the concept of solution of (P^0) and prove an existence and uniqueness result.

DEFINITION 3. *We shall say that* (P^0) *has a solution* $\boldsymbol{y}$ *iff*

i) $\boldsymbol{y} \in A^Q(V(r), S^0) \quad \forall r \in N$;

ii) $[\boldsymbol{y}]_r = \boldsymbol{y} + I^Q(V(r), S^0)$ *provides an r-solution to* (P^0) $\forall r \in N$.
We shall say that a solution $\boldsymbol{y}$ *to* (P^0) *is unique iff*

$$\boldsymbol{z} \text{ soluzion of } (P^0) \Rightarrow \boldsymbol{y} - \boldsymbol{z} \in I^Q(V(r), S^0) \qquad \forall r \in N .$$

It is now easy to prove

THEOREM 4. (Existence and uniqueness theorem) *Suppose given* (P^0): *it has a unique solution if* (S_k) *holds* $\forall k$, *big enough.*

Proof. By Theorem 2, $\boldsymbol{y} = (y_\nu)_{\nu \in N}$ with

$$y_\nu(x) = \exp\left(\int_\alpha^x s_\nu(u)\,du\right)\left[\beta_\nu + \int_\alpha^x t_\nu(v) \exp\left(-\int_\alpha^v s_\nu(u)\,du\right) dv\right]$$

is a solution. Moreover, if $\boldsymbol{z}$ is another solution, then by Theorem 3 we get $[\boldsymbol{y}]_r = [\boldsymbol{z}]_r$ is r is big enough, so, since $(I^Q(V(r), S^0))_{r \in N}$ is a decreasing sequence of sets, we have $\boldsymbol{y} - \boldsymbol{z} \in I^Q(V(r), S^0) \cdot \forall r \in N$.

Consider now the Cauchy problem

$$(R) \begin{cases} \boldsymbol{b}y' = \boldsymbol{s}y + \boldsymbol{t} \\ y(\alpha) = \beta \end{cases}$$

with $\boldsymbol{b}, \boldsymbol{s}, \boldsymbol{t} \in S^0$, $\beta \in C^N$, $\alpha \in R$.

DEFINITION 4. *We call r–solution of* (R), $r \in N$, *an element* $y \in A_r$ *such that*

$$(R_r) \begin{cases} [\boldsymbol{b}]_{r-1} D_{r,r-1} y = [\boldsymbol{s}]_{r-1} \gamma_{r,r-1} y + [\boldsymbol{t}]_{r-1} \\ \boldsymbol{z}(\alpha) - \beta \in I^Q(V(r-1), S^0) \end{cases}$$

for a suitable $\boldsymbol{z} \in A^Q(V(r), S^0)$, $y = [\boldsymbol{z}]_r$.
Solutions are defined in an analogous manner as before.

Introduce now the hypothesis

$$(H)\ \forall r \in N\ \exists \omega^{[r]} \in (V(r), S^0) \quad \text{such that} \quad \operatorname{Arg}(b_\nu(x) + \omega_\nu^{[r]}(x)) \neq \pi \qquad \forall \nu \in N, \quad x \in R .$$

We can now state and prove an existence result for (R).

THEOREM 5. *Suppose given* (R) *with* $\boldsymbol{b}$ *satisfying* (H): *then* (R) *has an r-solution* $\forall r \in N$. *If* $\omega^{[r]} = \omega$ $\forall r \in N$, (R) *has a solution.*

Proof. Introduce the 'Cauchy problem'.

$$(\mathrm{R}\,,r)\quad \begin{cases} \boldsymbol{u}' = \dfrac{\boldsymbol{s}}{\boldsymbol{b}+\omega^{[r]}}\,\boldsymbol{u} + \dfrac{\boldsymbol{t}}{\boldsymbol{b}+\omega^{[r]}} \\ \boldsymbol{u}(\alpha) = \beta \end{cases}$$

$\forall r \in \mathrm{N}$ the algebra $\mathrm{A}^{\mathrm{Q}}(\mathrm{V}(r), \mathrm{S}^0)$ contains $(\boldsymbol{b}+\omega^{[r]})^{-1}$ by Theorem 1, iii) ($f(z) = z^{-1}$), so by Theorem 2 the unique solution $\boldsymbol{u}^{[r]}$ of $(\mathrm{R}\,,r)$ in $\mathrm{W}(\mathrm{R})$ belongs to $\mathrm{A}^{\mathrm{Q}}(\mathrm{V}(r), \mathrm{S}^0)$.

Put now $y = \boldsymbol{u}^{[r]} + \mathrm{I}^{\mathrm{Q}}(\mathrm{V}(r), \mathrm{S}^0)$: y is an r-solution of (R).

In fact, $\boldsymbol{b}\cdot(\boldsymbol{u}^{[r]})' - \boldsymbol{s}\boldsymbol{u}^{[r]} - \boldsymbol{t} = (\boldsymbol{b}+\omega^{[r]})(\boldsymbol{u}^{[r]})' - \boldsymbol{s}\boldsymbol{u}^{[r]} - \boldsymbol{t} - \omega^{[r]}(\boldsymbol{u}^{[r]})' = -\omega^{[r]}(\boldsymbol{u}^{[r]})' \in \mathrm{I}^{\mathrm{Q}}(\mathrm{V}(r-1), \mathrm{S}^0)$ and

$$[\boldsymbol{u}^{[r]}]_r = y\,,\ \boldsymbol{u}^{[r]}(\alpha) - \beta = 0 \in \mathrm{I}^{\mathrm{Q}}(\mathrm{V}(r-1), \mathrm{S}^0).$$

If $\omega^{[r]}$ does not depend on r then $\boldsymbol{u} \in \mathrm{A}^{\mathrm{Q}}(\mathrm{V}(r), \mathrm{S}^0)$ $\forall r \in \mathrm{N}$ and is clearly a solution.

BIBLIOGRAPHY

[1] Elemer E. ROSINGER (1978). – *Distributions and Nonlinear Partial Differential Equations*, « Lecture Notes in Mathematics » *684*, Berlin, Springer-Verlag. XI–146.