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On bounded and total biorthogonal systems spanning given subspaces

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Analisi matematica. — *On bounded and total biorthogonal systems spanning given subspaces.* Nota [(*)] di PAOLO TERENZI [(**)], presentata dal Socio L. AMERIO.

RIASSUNTO. — Siano Y e Z due sottospazi quasi complementari di uno spazio di Banach separabile B. È noto (Vinokurov) che B ha una M-base unione di una M-base di Y e di una M-base di Z; inoltre è noto (Milman) che, se $\{y_n\}$ è una M-base di Y, esiste una successione $\{z_n\}$ di Z tale che $\{y_n\} \cup \{z_n\}$ sia una M-base di B.

Recentemente Ovsepian-Pelczynski, dando una risposta affermativa ad un problema da lungo tempo irrisolto, hanno dimostrato che B ha sempre una M-base uniformemente minimale.

Tale risultato pone allora la questione se sia possibile estendere alle M-basi uniformemente minimali il Teorema di Vinokurov e quello di Milman. Si dimostra, nel presente lavoro, che tale estensione non è possibile; anzi, se $\{y_n\}$ è una successione completa in Y, si dimostra che in generale non esiste una successione infinita $\{z_n\}$ di Z, tale che $\{y_n\} \cup \{z_n\}$ sia uniformemente minimale, anche nel caso di $\{y_n\}$ basica.

§ 1. INTRODUCTION

Notations, definitions and recalls are reported in § 2.

Let Y and Z be two quasi complementary subspaces of a separable Banach space B. It is known (Vinokurov) that B has an M-basis union of an M-basis of Y and of an M-basis of Z; moreover, if $\{y_n\}$ is an M-basis of Y, Milman stated that it is possible to extend $\{y_n\}$ to an M-basis of B by means of a sequence of Z. On the other hand a recent important result of Ovsepian-Pelczynsky stated the existence of an uniformly minimal M-basis for B. This raises the problem if it is possible to extend to the uniformly minimal M-bases the results of Vinokurov and Milman.

Then in § 3 we study if B has an uniformly minimal M-basis, union of an M-basis of Y and of an M-basis of Z, and we find that this is not in general possible; indeed, what's more, we prove that, if $\{y_n\}$ is an uniformly minimal sequence complete in Y, it is not possible in general to have an uniformly minimal sequence $\{y_n\} \cup \{z_n\}$ by means of an infinite sequence of Z. Therefore also Milman's theorem cannot be extended to the uniformly minimal M-bases.

In § 4 we are concerned with the extension of minimal and uniformly minimal sequences. In particular we point out that, if $\{y_n, h_n\}_{n\geq 1}$ is a biorthogonal system of B, with $\|y_n\| \cdot \|h_n\| \leq M < +\infty$ $\forall n \geq 1$ and with $\{y_n\}_{n\geq 1}$

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not complete in B, in general it is not possible to choose $y_0 \in$ B and $\{g_n\}_{n \geq 0} \subset$ B′ so that $\{y_n, g_n\}_{n \geq 0}$ is a biorthogonal system, with $\sup\{\|y_n\| \cdot \|g_n\|; n \geq 1\} < 2$M. We give also a proof of Milman's theorem.

In § 5 we define the M-bibasic systems and we study the extension of biorthogonal systems.

Finally in § 6 we raise a few problems, connected with the above questions.

§ 2. NOTATIONS, DEFINITIONS AND RECALLS

Theorems are enumerated by Roman figures and theorems of recalls by starred Roman figures. We use $\{n\}$ for the sequence of natural numbers, R^+ for the positive real semiaxis, $\mathscr{C}$ for the complex field, B for a separable Banach space and B′ for the dual of B.

Let $\{x_n\}$ be a sequence of B, then span $\{x_n\}$ is the linear manifold spanned by $\{x_n\}$, while $[x_n]$ is the closure of span $\{x_n\}$.

Let $\{x_n\} \subset$ B, we say that $\{x_n\}$ is:

a) *overfilling* if $[x_n] = [x_{n_k}]$ $\forall$ infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$;

b) *minimal* if $x_m \notin [x_n]_{n \neq m}$, $\forall m$;

c) *uniformly minimal* if $\inf_m \{\inf\{\|x_m + x\|; x \in \text{span}\{x_n\}_{n \neq m}\}\} > 0$.

Let $\{x_n\} \subset$ B and $\{f_n\} \subset$ B′, we say that $\{x_n, f_n\}$ is a

d) *biorthogonal system* if $f_m(x_n) = \delta_{mn}$ $\forall m$ and n;

e) *bounded biorthogonal system* if $f_m(x_n) = \delta_{mn}$ and $\|f_n\| \cdot \|x_n\| \leq \leq \text{M} < +\infty$, $\forall m$ and n.

It is well known ([3] and [2], see also [8] p. 54 and [7] p. 165) that: $\{x_n\}$ (*uniformly*) *minimal* $\Longleftrightarrow \exists \{f_n\} \subset$ B′ *with* $\{x_n, f_n|_{[x_k]}\}$ (*bounded*) *biorthogonal system*.

Let $\{x_n, f_n\}$ be a biorthogonal system, we recall that

f) $\{x_n\}$ is M-*basis* of B if $[f_n]$ is *total on* $[x_n]$ (that is $[f_n]_\perp \cap [x_n] = \{0\}$, where $[f_n]_\perp = \{x \in \text{B}; f_n(x) = 0 \ \forall n\}$) and if $[x_n] = $ B;

g) $\{x_n\}$ is *basis* of B if $x = \sum_1^\infty {}_n f_n(x) x_n$ $\forall x \in$ B;

h) $\{x_n\}$ is M-*basic* (*basic*) *sequence* if $\{x_n\}$ is M-basis (basis) of $[x_n]$.

Moreover we say that two subspaces Y and Z of B are *quasi complementary* if Y $\cap$ Z $= \{0\}$ and Y $+$ Z is dense in B.

About the uniformly minimal M-bases we recall

I*. (Pelczynsky [6]) *For every* $\varepsilon > 0$ B *has a total biorthogonal system* $\{y_n, h_n\}$ *with* $\{y_n\}$ *complete in* B *and* $\|y_n\| \cdot \|h_n\| < 1 + \varepsilon$ $\forall n$.

The same result had been found by Ovsepian-Pelczynski in a preceding Note [5], with the weaker condition that $\|y_n\| \cdot \|h_n\| < \text{M} < +\infty$ $\forall n$.

About the M-bases spanning given subspaces we recall

II*. (Vinokurov [12], see also [9] p. 187) *Let* Y *and* Z *be two quasi complementary subspaces of* B, *then* $\Rightarrow$ B *has an* M-*basis union of an* M-*basis of* Y *and of an* M-*basis of* Z.

Finally, about the extension of M-bases, we recall

III*. (Milman [4] p. 121). *Let* Y *and* Z *be two quasi complementary subspaces of* B *and let* $\{y_n\}$ *be an* M-*basis of* Y, *then* $\Rightarrow$B *has an* M-*basis* $\{y_n\} \cup \{z_n\}$, *where* $\{z_n\}$ *is a sequence of* Z.

§ 3. Uniformly minimal M-bases spanning given subspaces

In this paragraph we ask if Theorem II* keeps true for the uniformly minimal M-bases. More generally we consider the following question:

Let Y and Z be two quasi complementary subspaces of B, does it exist an M-basis $\{y_n\}$ of Y which is extendible to an uniformly minimal M-basis of B by means of a sequence of Z?

By next example we solve this question in the negative; then it follows that both Theorems II* and III* cannot be extended.

Example I. $\exists$ *a separable Banach space* B_1, *with two quasi complementary infinite dimensional subspaces* Y *and* Z, *so that, if* $\{y_n\}$ *and* $\{z_n\}$ *are two infinite sequences of* Y *and* Z *respectively, with* $[y_n] = Y$ *or with* $[z_n] = Z$, *then* $\{y_n\} \cup \{z_n\}$ *cannot be uniformly minimal.*

Proof. Let $\{v_n\} \cup \{w_n\}$ be a linearly independent sequence of elements of a linear space and let us set

$$(1)\qquad \left\| \sum_1^m{}_n (\alpha_n v_n + \beta_n w_n) \right\| = \sum_1^m{}_n \left(|\alpha_n + \beta_n| \left(1 - \frac{1}{2^n}\right) + \frac{|\alpha_n| + |\beta_n|}{2^n} \right), \quad \forall \{\alpha_n, \beta_n\}_{n=1}^m \subset \mathscr{C}.$$

If $\sum_1^m{}_n (\alpha_n v_n + \beta_n w_n)$ (where some α_n or β_n can be $= 0$) is the general element of span $\{v_n\}$ + span $\{w_n\}$, $\forall\lambda \cup \{\alpha_n, \alpha_n', \beta_n, \beta_n'\}_{n=1}^m \subset \mathscr{C}$ by (1) it immediately follows that

$$\left\| \sum_1^m{}_n (\alpha_n v_n + \beta_n w_n) \right\| = 0 \Longleftrightarrow \alpha_n = \beta_n = 0$$

$$\text{for} \quad 1 \leq n \leq m \Longleftrightarrow \sum_1^m{}_n (\alpha_n v_n + \beta_n w_n) = 0 ;$$

$$\left\| \sum_1^m{}_n (\lambda\alpha_n v_n + \lambda\beta_n w_n) \right\| = |\lambda| \cdot \left\| \sum_1^m{}_n (\alpha_n v_n + \beta_n w_n) \right\| ;$$

$$\left\| \sum_1^m{}_n (\alpha_n v_n + \beta_n w_n) \right\| \leq \left\| \sum_1^m{}_n (\alpha_n - \alpha_n') v_n + (\beta_n - \beta_n') w_n) \right\| + \\ + \left\| \sum_1^m{}_n (\alpha_n' v_n + \beta_n' w_n) \right\| .$$

Therefore (1) is a norm, hence (we can think $\{v_n\} \cup \{w_n\}$ in $C_{0 \vdash\!\dashv 1}$) let us set

$$(2) \qquad Y = [v_n] \quad , \quad Z = [w_n] \quad , \quad B_1 = [\{v_n\} \cup \{w_n\}] .$$

Moreover by (1) we have that

$$\left\| \sum_1^m{}_n (\alpha_n v_n + \beta_n w_n) \right\| \geq \sum_1^m{}_n \frac{|\alpha_n| + |\beta_n|}{2^n} , \qquad \forall \{\alpha_n , \beta_n\}_{n=1}^m \subset \mathscr{C}.$$

Therefore ([1], see also [8] p. 54) by (1) and (2) we have that

$$(3) \quad \begin{cases} \{v_n\} \cup \{w_n\} \text{ is minimal, moreover Y and Z are isometric to } l_1 , \\ \text{with } \{v_n\} \text{ natural basis of Y and } \{w_n\} \text{ natural basis of Z.} \end{cases}$$

By (3) it follows that

$$x \in Y \cap Z \Rightarrow x = \sum_1^\infty{}_n \alpha_n v_n = \sum_1^\infty{}_n \beta_n w_n \Rightarrow$$

$$\sum_1^\infty{}_n (\alpha_n v_n - \beta_n w_n) = 0 \Rightarrow \alpha_n = \beta_n = 0 \ \forall n \Rightarrow x = 0.$$

Hence, by (2), Y and Z are two quasi complementary subspaces of B_1. Let now $\{x_n\} \subset B_1$ so that

$$(4) \quad \begin{cases} \{x_n\} = \{y_n\} \cup \{z_n\} , \text{ with } \|x_n\| = 1 \ \forall n, \text{ moreover } [y_n] = Y \\ \text{and } \{z_n\} \text{ is an infinite sequence of Z.} \end{cases}$$

Let us fix $\bar{m} \in \{n\}$, we shall prove tha $\exists \bar{n} \in \{n\}$ so that

$$(5) \qquad \inf \{\|z_{\bar{n}} + x\| ; x \in \operatorname{span} \{z_n\}_{n \neq \bar{n}} + \operatorname{span} \{y_n\}\} < 1/2^{\bar{m}} .$$

By (3) and (4) we have that

$$(6) \qquad z_n = \sum_1^\infty{}_k \alpha_{nk} w_k , \qquad \text{with} \quad \sum_1^\infty{}_k |\alpha_{nk}| = 1 , \forall n.$$

By (6) $|\alpha_{nk}| \leq 1 \ \forall n$ and k; hence $\exists$ an infinite subsequence $\{n_i\}$ of $\{n\}$ so that

$$\lim_{i \to \infty} \alpha_{n_i k} = \alpha_k , \qquad \text{for} \quad 1 \leq k \leq \bar{m} + 2.$$

Therefore $\exists \bar{i} \in \{n\}$ so that, setting $n_{\bar{i}} = \bar{n}$ and $n_{\bar{i}+1} = \bar{n} + \bar{p}$, we have that

$$(7) \qquad \sum_1^{\bar{m}+2}{}_k |\alpha_{\bar{n}k} - \alpha_{\bar{n}+\bar{p},k}| < \frac{1}{2^{\bar{m}+1}} .$$

On the other hand, by (4), span $\{y_n\}$ is dense in Y; consequently by (1), (3), (4), (6) and (7) we have that

$$\inf \{ \|z_{\bar{n}} + x\| ; x \in \operatorname{span} \{z_n\}_{n \neq \bar{n}} + \operatorname{span} \{y_n\}\} \leq$$

$$\leq \left\| z_{\bar{n}} - z_{\bar{n}+\bar{p}} - \sum_{\bar{m}+3}^\infty{}_k \alpha_{\bar{n}k} v_k + \sum_{\bar{m}+3}^\infty{}_k \alpha_{\bar{n}+\bar{p},k} v_k \right\| = \left\| \sum_1^{\bar{m}+2}{}_k (\alpha_{\bar{n}k} - \alpha_{\bar{n}+\bar{p},k}) w_k + \right.$$

$$+ \sum_{\bar{m}+3}^{\infty}{}_k \alpha_{\bar{n}k} (w_k - v_k) + \sum_{\bar{m}+3}^{\infty}{}_k \alpha_{\bar{n}+\bar{p},k} (v_k - w_k) \Big\| \leq \Big\| \sum_{1}^{\bar{m}+2}{}_k (\alpha_{\bar{n}k} - \alpha_{\bar{n}+\bar{p},k}) w_k \Big\| +$$

$$+ \Big\| \sum_{\bar{m}+3}^{\infty}{}_k \alpha_{\bar{n}k} (w_k - v_k) \Big\| + \Big\| \sum_{\bar{m}+3}^{\infty}{}_k \alpha_{\bar{n}+\bar{p},k} (v_k - w_k) \Big\| = \sum_{1}^{m+2}{}_k | \alpha_{\bar{n}k} - \alpha_{\bar{n}+\bar{p},k} | +$$

$$+ \sum_{\bar{m}+3}^{\infty}{}_k \frac{|\alpha_{\bar{n}k}|}{2^{k-1}} + \sum_{\bar{m}+3}^{\infty}{}_k \frac{|\alpha_{\bar{n}+\bar{p},k}|}{2^{k-1}} < \frac{1}{2^{\bar{m}+1}} + \frac{1}{2^{\bar{m}+2}} \left(\sum_{\bar{m}+3}^{\infty}{}_k |\alpha_{\bar{n}k}| + \sum_{\bar{m}+3}^{\infty}{}_k |\alpha_{\bar{n}+\bar{p},k}| \right) \leq$$

$$\leq \frac{1}{2^{\bar{m}+1}} + \frac{1}{2^{\bar{m}+2}} 2 = \frac{1}{2^{\bar{m}}} .$$

That is (5) is proved, consequently $\{x_n\}$ is never uniformly minimal, which completes the proof of example I.

§ 4. Extension of minimal and uniformly minimal sequences

Firstly we consider the extension of minimal sequences, hence theorem III* of § 2.

It is not possible to improve this theorem, with the further condition of $\{z_n\}$ complete in Z. Indeed, if it is not $Y + Z = B$, Singer pointed out ([9], p. 186) that $\exists$ a particulr M-basis $\{y_n\}$ of Y, so that it is never possible to have $\{z_n\}$ complete in Z. Moreover the author [10] proved that, if Y is an infinite dimensional and codimensional subspace of B and if $\{y_n\}$ is an M-basis of Y, then $\exists$ a subspace Z of B, quasi complementary with Y, so that, in the Banach space B/Z, $\{y_n + Z\}$ is overfilling.

Milman stated Theorem III* without proof. We wish now to give a proof, precisely we prove that

I. *Let $\{y_n\}$ be a minimal sequence of* B *and let* Z *be a subspace of* B *quasi complementary with* $[y_n]$, *then*: $\Rightarrow \exists$ *a sequence* $\{z_n\}$ *of* Z *so that* $\{w_n\} = \{y_1, z_1, y_2, z_2, \cdots\}$ *is minimal and complete in* B, *with* $\bigcap_{m=1}^{\infty} [w_n]_{n>m} \subseteq [y_n]$.

Proof. Let $\{v_n\} \subset B$ and $\{\varepsilon_n\} \subset R^+$ so that

(8) $\quad \forall \{u_n\} \subset B$ with $\| u_n - v_n \| < \varepsilon_n \, \forall n$, $\{u_n\}$ is complete in B.

Moreover let us set

$$Y_n = [y_k]_{k \geq n} \quad \text{and} \quad B_n = B/Y_n, \quad \forall n.$$

We shall leave out the trivial case of Z finite dimensional subspace of B.

By hypothesis $\exists \{x_{1n}\}_{n \geq 1} \subset B$ so that

(9) $\quad \{x_{1n} + Y_1\}$ is M-basis of B_1, with $\{x_{1n}\} \subset Z$.

Then $\exists p_1 \in \{n\}$ and $u_1 \in B$ so that

$$\| u_1 - v_1 \| < \varepsilon_1 \,, \quad \text{with } u_1 \in \operatorname{span}\{y_n\} + \operatorname{span}\{x_{1n}\}_{n=1}^{p_1} . \tag{10}$$

By hypothesis and by (9) $\{y_1 + Y_2\} \cup \{x_{1n} + Y_2\}_{n=1}^{p_1}$ is a linearly independent sequence of B_2, hence $\exists \{x_{2n}\}_{n\geq 1} \subset B$ so that

$$\begin{cases} \{y_1 + Y_2\} \cup \{x_{1n} + Y_2\}_{n=1}^{p_1} \cup \{x_{2n} + Y_2\}_{n\geq 1} \quad \text{is M-basis of } B_2 \,, \\ \text{with } \{x_{2n}\}_{n\geq 1} \subset \operatorname{span}\{x_{1n}\}_{n>p_1} . \end{cases} \tag{11}$$

Then $\exists p_2 \in \{n\}$ and $u_2 \in B$ so that

$$\| u_2 - v_2 \| < \varepsilon_2 \,, \quad \text{with } u_2 \in \operatorname{span}\{y_n\} + \sum_1^2{}_i \operatorname{span}\{x_{in}\}_{n=1}^{p_i} . \tag{12}$$

Now $\{y_n + Y_3\}_{n=1}^{2} \cup \{\bigcup_{i=1}^{2} \{x_{in} + Y_3\}_{n=1}^{p_i}\}$ is linearly independent in B_3 then we can extend this sequence to an M-basis of B_3, by means of a sequence $\{x_{3n} + Y_3\}_{n\geq 1}$, with $\{x_{3n}\} \subset \operatorname{span}\{x_{2n}\}_{n>p_2}$.

So proceeding, by (9), (10), (11) and (12) we find $\{x_n\}$ and $\{u_n\}$ in B so that

$$\begin{cases} \{x_n\} = \bigcup_{i=1}^{\infty} \{x_{in}\}_{n=1}^{p_i} \,, \quad \text{where, } \forall m \,, \\ \{y_n + Y_m\}_{i=1}^{m-1} \cup \left\{ \bigcup_{i=1}^{m-1} \{x_{in} + Y_m\}_{n=1}^{p_i} \right\} \cup \{x_{mn} + Y_n\}_{n\geq 1} \text{ is M-basis of} \\ B_m \,, \text{ with } \{x_{mn}\}_{n\geq 1} \subset \operatorname{span}\{x_{m-1,n}\}_{n>p_{m-1}} ; \text{ moreover} \,, \quad \forall m \,, \\ \| u_m - v_m \| < \varepsilon_m \text{ and } \{u_n\}_{n=1}^{m-1} \subset \operatorname{span}\{y_n\} + \sum_1^{m-1}{}_i \operatorname{span}\{x_{in}\}_{n=1}^{p_i} . \end{cases} \tag{13}$$

By (8), (9) and (13) it follows that

$$\{y_n\} \cup \{x_n\} \quad \text{is complete in } B \,, \text{ with } \quad \{x_n\} \subset Z . \tag{14}$$

Moreover by (13), $\forall m$, we have that $y_m \notin [\{y_n\}_{n\neq m} \cup \{x_n\}]$, hence $\exists (h_n\} \subset B'$ so that

$$\{y_n \,, h_n\} \quad \text{is a biorthogonal system, with } \quad \{x_n\} \subset [h_n]_\perp . \tag{15}$$

By (14) $\exists \{z_n\} \subset B$ so that

$$\{z_n + Y_1\} \quad \text{is M-basis of } B_1 \,, \text{ with } \quad \{z_n\} \subset \operatorname{span}\{x_n\} . \tag{16}$$

By (16) $\exists \{G_n\} \subset B_1'$ so that $\{z_n + Y_1 \,, G_n\}$ is a biorthogonal system; therefore, if we set, $\forall n$, $g_n(x) = G_n(x + Y_1)$ $\forall x \in B$, it follows that

$$\{z_n \,, g_n\} \quad \text{is a biorthogonal system, with } \quad \{g_n\} \subset Y_1^\perp . \tag{17}$$

Consequently by (15), (16) and (17) it follows that

$$(18)\quad \begin{cases} \{w_n\} = \{y_n\} \cup \{z_n\} \text{ is complete in B, with } \{y_n, h_n\} \cup \{z_n, g_n\} \\ \text{biorthogonal system, moreover } \displaystyle\bigcap_{m=1}^{\infty} [w_n]_{n>m} \subseteq Y_1 = [y_n] . \end{cases}$$

This completes the proof of Theorem I.

We remark that, if in Theorem I $\{y_n\}$ is M-basic, then in (18) $[h_n]$ is total on $[y_n]$; therefore, if $\bar{x} \in \bigcap_{m=1}^{\infty} [w_n]_{n>m}$, by (18) $h_n(\bar{x}) = g_n(\bar{x}) = 0\ \forall n$ and $\bar{x} \in [y_n]$, hence $\bar{x} = 0$, that is $\{w_n\}$ is M-basis of B ([2], see also [7] p. 171), consequently we have Theorem III*.

Let us now consider the extension of uniformly minimal sequences. This is a more difficult problem than for the minimal sequences; indeed, by example I, we have already seen that Theorem III* does not keep true, also without the condition of $\{y_n\} \cup \{z_n\}$ complete in B. Moreover also the extension by an only element presents difficulties for an uniformly minimal sequence, indeed

Example II. *l_1 has a biorthogonal system $\{y_n, h_n\}_{n\geq 1}$ with $\|y_n\| = \|h_n\| = 1$ $\forall n \geq 1$ and $\{y_n\}_{n\geq 1}$ not complete in l_1, so that, if $\{y_n, g_n\}_{n\geq 0}$ is a biorthogonal system of l_1, it follows that* $\sup_n \|g_n\| \geq 2$.

Proof. Let $\{x_n\}_{n\geq 0}$ be the natural basis of l_1 and let us set

$$(19)\qquad y_n = (x_n + x_0)/2 \qquad \forall n \geq 1 .$$

Then $\forall \{\alpha_n\}_{n=1}^{m} \subset \mathscr{C}$ we have that

$$(20)\qquad \left\| x_0 + \sum_1^m {}_n\, \alpha_n y_n \right\| = \left\| x_0 \left(1 + \sum_1^m {}_n \frac{\alpha_n}{2} \right) + \sum_1^m {}_n \frac{\alpha_n}{2} x_n \right\| =$$
$$= \left| 1 + \sum_1^m {}_n \frac{\alpha_n}{2} \right| + \sum_1^m {}_n \frac{|\alpha_n|}{2} \geq 1 .$$

By (20) $x_0 \notin [y_n]_{n\geq 1}$ hence by (19) it follows that

(21) span $\{x_0\}$ and $[y_n]_{n\geq 1}$ are two complementary subspaces of l_1.

Moreover, $\forall m$ and $\forall \{\alpha_n\}_{n(\neq m)=1}^{p} \subset \mathscr{C}$, by (19) it follows that

$$\left\| y_m + \sum_1^p {}_{n(\neq m)}\, \alpha_n y_n \right\| = \left\| x_m \cdot \tfrac{1}{2} + x_0 \left(\tfrac{1}{2} + \sum_1^p {}_{n(\neq m)} \frac{\alpha_n}{2} \right) + \sum_1^p {}_{n(\neq m)} \frac{\alpha_n}{2} x_n \right\| =$$
$$= \tfrac{1}{2} + \left| \tfrac{1}{2} + \sum_1^p {}_{n(\neq m)} \frac{\alpha_n}{2} \right| + \sum_1^p {}_{n(\neq m)} \frac{|\alpha_n|}{2} \geq 1 .$$

Consequently, by Hahn Banach Theorem, $\exists \{h_n\}_{n\geq 1} \subset l_1'$ so that $\{y_n, h_n\}_{n\geq 1}$ is a biorthogonal system, with $\|y_n\| = \|h_n\| = 1\ \ \forall n \geq 1$.

Let now $y_0 \in l_1$ and $\{g_n\}_{n\geq 0} \subset l_1'$ so that

$$\{y_n, g_n\}_{n\geq 0} \text{ is a biorthogonal system.} \tag{22}$$

By (21) we have that

$$y_0 = \bar{\alpha}x_0 + \tilde{y}, \quad \text{with } \bar{\alpha} \neq 0 \text{ and } \tilde{y} \in [y_n]_{n\geq 1}. \tag{23}$$

Let us fix $\varepsilon > 0$ and let $\{\alpha_n\}_{n=1}^{p} \subset \mathscr{C}$ so that

$$\left\| \tilde{y} - \sum_1^p {}_n\, \alpha_n y_n \right\| < 2\varepsilon\, |\bar{\alpha}|. \tag{24}$$

By (19), (23) and (24) it follows that

$$\left\| y_{p+1} - \left(y_0 - \sum_1^p {}_n\, \alpha_n y_n\right) \frac{1}{2\bar{\alpha}} \right\| = \left\| y_{p+1} - \left(\bar{\alpha}x_0 + \tilde{y} - \sum_1^p {}_n\, \alpha_n y_n\right) \frac{1}{2\bar{\alpha}} \right\| \leq$$

$$\leq \left\| y_{p+1} - \frac{x_0}{2} \right\| + \left\| \tilde{y} - \sum_1^m {}_n\, \alpha_n y_n \right\| \frac{1}{2\,|\bar{\alpha}|} < \tfrac{1}{2} + 2\varepsilon\,|\bar{\alpha}|\, \frac{1}{2\,|\bar{\alpha}|} = \tfrac{1}{2} + \varepsilon.$$

That is $\inf_m \{\inf \{\|y_m + y\|\,; y \in \operatorname{span}\{y_n\}_{n(\neq m)=0}^{\infty}\}\} \leq \frac{1}{2}$; consequently, by (22), $\sup_m \|g_m\| \geq 2$. This completes the proof of example II.

We shall continue our considerations on the extension of uniformly minimal sequences in § 6.

§ 5. M-BIBASIC SYSTEMS AND EXTENSION OF BIORTHOGONAL SYSTEMS

Let us consider a biorthogonal system $\{y_n, h_n\}$ of B, we shall say that

(D_1) $\{y_n, h_n\}$ is *extendible* if $\exists\, \{z_n\} \subset$ B and $\{g_n\} \subset$ B′ so that $\{y_n, h_n\} \cup \cup \{z_n, g_n\}$ is a biorthogonal system with $\{y_n\} \cup \{z_n\}$ complete in B.

We point out that

$$\{y_n, h_n\} \text{ is extendible} \Rightarrow \{h_n\} \text{ is M-basic.} \tag{25}$$

In fact, if Q is the canonical mapping of B into B″, in (D_1) we have that $\{h_n, Q(y_n)\} \cup \{g_n, Q(z_n)\}$ is a biorthogonal system of B′, with $[\{Q(y_n)\} \cup \cup \{Q(z_n)\}]$ total on $[\{h_n\} \cup \{g_n\}]$.

Moreover we recall that, if $\{u_n\}$ is a minimal sequence not complete in B, $\exists\, \{h_n\} \subset$ B′ so that $\{y_n, h_n\}$ is a biorthogonal system, but not extendible ([9] p. 184, see also [11] corollary I).

We also point out that, if $\{y_n, h_n\}$ is a biorthogonal system, it is possible that $\{h_n\}$ is M-basic and $\{y_n\}$ not (for example, if $\{y_n\}$ is complete in B but not M-basic, by (25)); moreover $\{y_n\}$ can be M-basic and $\{h_n\}$ not (for example, if $\{y_n\}_{n\geq 0}$ is M-basis of B, with $\{y_n, f_n\}_{n\geq 0}$ biorthogonal system, setting $h_n = f_n + n\,\|f_n\|\, f_0\ \forall n \geq 1$, we have that $\{y_n, h_n\}_{n\geq 1}$ is a biorthogonal system, but $\{h_n\}$ is not M-basic, because $\lim_{n\to\infty} h_n/\|h_n\| = f_0/\|f_0\|$).

Therefore, if $\{y_n, h_n\}$ is a biorthogonal system, we shall say that

(D_2) $\{y_n, h_n\}$ is M-*bibasic* if both $\{y_n\}$ and $\{h_n\}$ are M-basic.

By (25) and Theorem III* it follows that every M-basic sequence $\{y_n\}$ of B belongs to an M-bibasic system $\{y_n, h_n\}$.

Moreover, by (D_1) and (D_2), we shall say that

(D_3) $\{y_n, h_n\}$ is M-*extendible* if is extendible to an M-bibasic system complete in B.

By (D_3) it is obviuos that an M-extendible system is M-bibasic, but this necessary condition is not in general sufficient, indeed:

Example III. c_0 has an M-*bibasic system which is not extendible.*

Proof. Suppose that

(26) $\{x_n\}_{n\geq 0}$ is the natural basis of c_0, with $\{x_n, f_n\}_{n\geq 0}$ biorthogonal system.

Then let us set

(27) $$h_n = f_n + f_0 \qquad \forall n \geq 1 .$$

Suppose that for an $\bar{x} \in c_0$, $h_n(\bar{x}) = 0$ $\forall n$, by (27) $f_n(\bar{x}) = -f_0(\bar{x})$ $\forall n \geq 1$; on the other hand by (26) $\bar{x} = \sum_0^\infty {}_n f_n(\bar{x}) x_n$, hence $f_n(\bar{x}) = 0$ $\forall n \geq 0$, that is $\bar{x} = 0$. Consequently $[h_n]$ is total on c_0, therefore by (26) and (27) it follows that

(28) $\{x_n, h_n\}_{n\geq 1}$ is a biorthogonal system of c_0 but not extendible.

Now c_0' is isometric to l_1, then we can consider $\{f_n\}_{n\geq 0}$ as the natural basis of l_1, therefore by (19), (20), (26) and (27) it follows that

(29) $$f_0 \notin [h_n] .$$

Suppose that

(30) $$\bar{h} \in [h_n] \text{ with } \bar{h}(x_n) = 0 \quad \forall n \geq 1 .$$

By (27) and (30) $\bar{h} = \bar{\alpha} f_0 + \bar{f}$ with $\bar{f} \in [f_n]_{n\geq 1}$; now $\bar{f} = 0$ by (26) and (30), because $\bar{h}(x_n) = \bar{f}(x_n)$ for $n \geq 1$ and $\{f_n\}_{n\geq 1}$ is M-basic; hence $\bar{h} = \bar{\alpha} f_0$, that is $\bar{h} = 0$ by (29) and (30), whence $[x_n]_{n\geq 1}$ is total on $[h_n]$. Therefore $\{h_n\}$ is M-basic, which, by (26) and (28), completes the proof of example III.

Let us give now a few characterizations for (D_1), (D_2) and (D_3).

II. *Let $\{y_n, h_n\}$ be a biorthogonal system of* B, *then*

a) $\{y_n, h_n\}$ *is extendible* $\Longleftrightarrow [y_n] + [h_n]_\perp$ *is dense in* B.

b) $\{y_n, h_n\}$ *is* M-*bibasic* $\Longleftrightarrow [y_n]^\perp \cap [h_n] = \{0\}$ and $[y_n] \cap [h_n]_\perp = \{0\}$.

c) $\{y_n, h_n\}$ *is* M-*extendible* $\Longleftrightarrow \{y_n, h_n\}$ *is extendible with* $\{y_n\}$ M-*basic* $\Longleftrightarrow [y_n]$ *and* $[h_n]_\perp$ *are quasi complementary subspaces of* B.

Proof.:

a) $\Rightarrow$ is obvious, while $\Leftarrow$ follows by Theorem I.

b) It is obviuos.

c) It follows by (25) and by *a*) and b).

§ 6. OPEN PROBLEMS

The main open problem on the extension of uniformly minimal sequences is

Problem 1. Let $\{y_n\}$ be an uniformly minimal M-basic sequence of B, does it exist $\{z_n\} \subset$ B so that $\{y_n\} \cup \{z_n\}$ becomes an uniformly minimal M-basis of B?

A weaker version of this problem is

Problem 2. Let $\{y_n\}$ be an uniformly minimal M-basic sequence of B, does it exist $\{h_n\} \subset$ B$'$ so that $\{y_n, h_n\}$ becomes a bounded and M-extendible biorthogonal system?

We remark that, if $\{y_n, h_n\}$ is a bounded and M-extendible biorthogonal system of B and if $\{n_k\}$ and $\{n'_k\}$ are two infinite complementary subsequences of $\{n\}$, by propositions 1 of [5] it follows that both $\{y_{n_k}\}$ and $\{y_{n'k}\}$ are extendible to an uniformly minimal M-basis of B. This raises the following question, about a possible equivalence between problems 1 and 2.

Problem 3. Let $\{y_n, h_n\}$ be a bounded and M-extendible biorthogonal system of B, is $\{y_n\}$ extendible to an uniformly minimal M-basis of B?

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