Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Solutions with "precise" compact support of the $\bar{\partial}$-Problem in strictly pseudoconvex domains and some consequences

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **67** (1979), n.1-2, p. 81–86.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_67_1-2_81_0>

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Geometria. — *Solutions with "precise" compact support of the $\bar\partial$-Problem in strictly pseudoconvex domains and some consequences*
Nota (*) di MARIO LANDUCCI, presentata (**) dal Socio G. ZAPPA.

RIASSUNTO. — Si forniscono soluzioni con supporto compatto « preciso » del $\bar\partial$-problema in domini strettamente pseudoconvessi, e se ne deducono alcune conseguenze.

1. INTRODUCTION AND MOTIVATION OF THE PROBLEM

Let D a bounded strictly pseudoconvex domain in C^n, $n \geq 2$, with smooth boundary, i.e.

$$D = \{z \in U : \rho(z) < 0\}, \qquad U \supset \overline{D}$$

where ρ is a strictly plurisubharmonic C^∞ function near ∂D and grad $\rho \neq 0$ on ∂D.

If f is a $(0, q+1)$ $\bar\partial$-closed differential form with $C^k(\overline{D})$, $k \geq 1$, coefficients, then the $\bar\partial$-equation

$$\bar\partial u = f \tag{1}$$

has solutions $C^k(\overline{D})$ which satisfy the uniform estimates (see [3])

$$\| u \|_{k+\frac{1}{2}} \leq C_k(D) \| f \|_k \tag{2}$$

where

$$\| \cdot \|_k = \max_{|I| \leq k} \sup_D | D^I \cdot |$$

$(D^I$ any differentiation of order $\leq k)$

$$\| u \|_{k+\frac{1}{2}} = \max \left[\| u \|_k \; ; \; \max_{|I| = k} \sup_{z,z' \in D} \frac{D^I u(z) - D^I u(z')}{|z - z'|^{\frac{1}{2}}} \right]$$

and $C_k(D)$ is a numerical constant depending only on k and D; this, in particular, means that (1) has solutions which improve the regularity of g of an Holder exponent $\frac{1}{2}$.

If in addition f is compactly supported on $\overline{D}$ (that is supp$f \subseteq \overline{D}$) and $k = \infty$ thm in [1] implies that fixed any $D' \supset \overline{D}$ it is, consequently, possible to find solutions of (1) compactly supported in D'.

(*) Lavoro eseguito nell'ambito dell'attività del G.N.S.A.G.A del C.N.R.
(**) Pervenuta all'Accademia il 17 luglio 1979.

From above it naturally arises the question to know if it is possible to find solutions of (1) which conserve the same support of the data, when supp$f$ is a strictly pseudoconvex domain, and at the same time improve its regularity (i.e. it is possible to prove estimates of type (2) for these solutions).

The motivation of such a question lies on the fact that the knowledge of solutions with these properties allows

a) to give an "Hartogs-like theorem", with estimates, for $\bar\partial$-closed differential forms on a annolus difference of two strictly pseudoconvex domains (Proposition 1);

b) to characterize the boundary values of $\bar\partial$-closed forms (Proposition 2);

c) to reduce the study, in the differential case, of $\bar\partial_b$-operator on ∂D to the study of $\bar\partial$-operator on D (Proposition 3).

2. Statement of the main Theorem and sketch of the proof

The following theorem holds,

THEOREM. *Let f be a* $(0, q+1)$ *$\bar\partial$-closed form with* $C^{k-1}(\overline{D})$ *coefficients,* $\|f\|_k < \infty$ *and support of* $f \subseteq \overline{D}$. *Then if* $q < n-1$ (1) *has always a solution* $u \in C_0^{k+\frac{1}{2}}(\overline{D})$ (*the equality of supports verifies when support* $f = \overline{D}$) s.t.

$$\|u\|_{k+\frac{1}{2}} \leq C\, \|f\|_k$$

If $q = n-1$ *there exists* u *with the previous properties if and only if*

$$\int_{\mathbf{C}^n} hf \wedge dz_1 \wedge \cdots \wedge dz_n = 0$$

for every h *holomorphic in* D *and smooth on* $\overline{D}$.

Remarks:

A) We shall not examine the case $q = 0$ because in this case the theorem follows directly from [5] jointly with [6].

B) The additional hypothesis required in the case $q = n-1$ is effectively necessary: let us take in fact the Bochner-Martinelli kernel $K(z, w)$ and a function φ which takes values 1 near ∂D and 0 in a neighbourhood of w, then for every h holomorphic in D and continuous in D we have,

$$h(w) = \int_{z \in \partial D} h\varphi\, K(z, w) = \int_{z \in D} h\, \bar\partial(\varphi K)$$

Suppose $h(w) \neq 0$: as $\bar\partial(\varphi k)$ is compactly supported (K is $\bar\partial$-closed and smooth for $z \neq w$) if the theorem would be true, without additional hypothesis in the case $q = n-1$, this would imply (by Stokes' theorem) $h(w) = 0$. Absurd.

Sketch of the proof of the Theorem

Firstable we take an explicit particular solution of (1) (unfortunately not compactly supported) defined in $\mathbf{C}^n$ which is given by

$$(+) \qquad \tilde{u}(w) = \int_{\mathbf{C}^n} f \wedge \mathrm{K}_q(\zeta, w)$$

where K_q is the q-th Bochner-Martinelli-Koppelman kernel and we prove the required regularity for $\tilde{u}$.

The strategy is now to correct $\tilde{u}$ by a $\bar{\partial}$-exact for which coincides with $\tilde{u}$ near $\partial\mathrm{D}$ and outside D.

For this purpose we use the geometrical assumptions on D, that is the existence of a function $\Phi(\zeta, w) = \Sigma\Phi_i \cdot (\zeta_i - w_i)$, Henkin's function (see [4]), which is holomorphic in $\zeta \in \mathrm{U}$, U open neighbourhood of $\overline{\mathrm{D}}$, for fixed w in a neighbourhood of $\partial\mathrm{D}$, and whose zero set lies entirely outside D, when $w \in \mathrm{C}^n - \mathrm{D}$.

By the Φ_i's we then construct Cauchy-Fantappiè forms $\Omega_q(\zeta, w)$ s.t. $\mathrm{K}_q(\zeta, w) = \bar{\partial}_\zeta \Omega_q + \bar{\partial}_w \Omega_{q+1}$ for $q < n - 1$, $\mathrm{K}_{n-1}(\zeta, w) = \bar{\partial}_w \Omega_n +$ holom-func. in w and this imples, by (+), that

$$\tilde{u} = \bar{\partial} v \qquad \text{outside D and near } \partial\mathrm{D}$$

where

$$v(w) = \int_{\mathrm{D}} f \wedge \Omega_{q+1}(\zeta, w)$$

The second part of the proof consists of a careful study of the regularity of v in terms of f.

The theorem now follows taking a convenient estension of $v(w)$, $\mathrm{V}(w)$, to D which preserves the regularity of v (being $\partial\mathrm{D}$ smooth this is always possible): the differential form $u = \tilde{u} - \bar{\partial} v$ is the required solution.

3. SOME APPLICATIONS AND CONSEQUENCES

Let $\mathrm{D}_1 \supset\supset \mathrm{D}_2$ two strictly pseudoconvex domains with smooth boundaries and $\Omega = \mathrm{D}_1 - \mathrm{D}_2$.

PROPOSITION 1. *Let ψ a $(0, q+1)$, $q < n - 1$, differential form $\bar{\partial}$-closed and smooth in $\overline{\Omega}$*

Then there exists a $(0, q+1)$ differential form Ψ smooth and $\bar{\partial}$-closed in $\overline{\mathrm{D}}_1$ which extends ψ (i.e. $\Psi_{|\Omega} = \psi$) and such that

$$\|\Psi\|_{k-1+\frac{1}{2}} \leq \mathrm{C}_k \|\psi\|_{k,\Omega},$$

If $q = n - 1$, *the extension* Ψ *with the previous properties exists provided that*

$$\int_{\partial D_2} h\psi \wedge dz_1 \wedge \cdots \wedge dz_n = 0$$

for every h *holomorphic in* D_2 *and smooth in* $\overline{D}_2$.

Proof. Take a C^∞ extension of ψ, $\tilde{\psi}$, to D_1 preserving its regularity. Then it is sufficient to apply the theorem to the $\bar{\partial}$-problem

$$\bar{\partial} u = \bar{\partial} \tilde{\psi}$$

The differential form $\Psi = \psi - u$ satisfies the thesis.

Proposition 1 has the obvious consequence that the $\bar{\partial}$-problem

$$\bar{\partial}\alpha = \beta$$

with $\beta \in C^k(\overline{\Omega})$ and $\bar{\partial}\beta = 0$ on Ω, admits solutions α which satisfy

$$\| \alpha \|_{k-1+\frac{1}{2}} \leq C_k \| \beta \|_k$$

Definition. Let f a $(0, q+1)$ differential form C^k on ∂D (i.e. the restriction of a $(0, q+1)$ C^k differential form defined in a neighbourhood of ∂D) we shall say that

$$\bar{\partial}_b f = 0$$

if and only if for every θ $(n, n-q-2)$ differential form smooth in $\overline{D}$ and $\bar{\partial}$-closed we have,

$$(++) \qquad \int_{\partial D} f \wedge \theta = 0$$

We observe that if f is the boundary value of a $\bar{\partial}$-closed differential form then clearly $\bar{\partial}_b f = 0$ (it is sufficient to apply Stokes' theorem).

This condition, in the strictly pseudoconvex case, is also sufficient. In fact we have,

PROPOSITION 2. *Let* f *a* $C^k(\partial D)$ $(0, q)$ *differential form restriction to* ∂D *of* $\hat{f}$, *then it is the boundary value of a* $\bar{\partial}$-*closed differential form* F *if and only if* $\bar{\partial}_b f = 0$. *Furthermore the extension* F *satisfies*,

$$\| F \|_{k-1+\frac{1}{2}, D} \leq C \| \hat{f} \|_{k, \partial D} .$$

Line of the proof: The first step is to show that, as $\bar{\partial}_b f = 0$, we can then construct an extension $\tilde{F}$ of f to D s.t.

$$\bar{\partial} \tilde{F} = o(\rho^{k-1})$$

where ρ is the defining function of D.

The second step is to prove, with a similar argument used in the proof of the theorem, that in this case the $\bar\partial$-problem

$$\bar\partial u = \bar\partial \tilde{F}$$

admits $C^{k-1+\frac{1}{2}}$ compactly supported solutions s.t.

$$\| u \|_{k-1+\frac{1}{2}} \leq C_k \| \tilde{F} \|_k$$

Finally the differential form

$$F = \tilde{F} - u$$

completely satisfies the thesis.

Let us consider now the $\bar\partial_b$-equation on ∂D

$$(+++) \qquad \bar\partial_b u = f$$

(for the relative definitions see [2]) where f is a $(0, q+1)$ $q < n-1$, $C^k(\partial D)$ differential form which satisfies the necessary compatibility condition $\bar\partial_b f = 0$.

By Proposition 2 f is then the boundary value of a $\bar\partial$-closed form in D, say F.

Solving now in D the $\bar\partial$-equation,

$$\bar\partial U = F$$

it clearly follows that $u = U|_{\partial D}$ is a solution of $(+++)$.

In conclusion we have that,

PROPOSITION 3. *Let* D *a strictly pseudoconvex domain in* C^n, *with smooth boundary, and* f *a* $C^k(\partial D)$, $\bar\partial_b$-*closed differential form, restriction of* $\tilde{f}$ *to* ∂D. *Then the* $\bar\partial_b$-*equation*

$$\bar\partial_b u = f$$

admits $C^{k-1+\frac{1}{2}}(D)$ *solutions which satisfy the uniform estimates*

$$\| u \|_{k-1+\frac{1}{2}} \leq C_k \| \tilde{f} \|_k .$$

The proofs of the Theorem and propositions will appear in all the details elsewhere. (*)

(*) *Added in Proof*: In «Bulletin des Sciences Mathématiques» with the title, *Solutions with precise compact support of* $\bar\partial u = f$.

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