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On the asymptotic stability for abstract Volterra integro-differential equations

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Equazioni funzionali. — *On the asymptotic stability for abstract Volterra integro-differential equations.* Nota (*) di ANDREA SCHIAFFINO e ALBERTO TESEI, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Si danno condizioni per la stabilità asintotica della soluzione nulla di una classe di equazioni integro-differenziali di Volterra in spazi di Banach.

1. INTRODUCTION

We are concerned in the present paper with Liapunof stability and attractivity properties of the trivial solution of the linear problem:

$$\text{(P)} \qquad \begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = -\mathrm{A}u(t) + \displaystyle\int_0^t \mathrm{d}s\, k(t-s)\, u(s) & (t > 0) \\ u(0) = u_0 ; \end{cases}$$

here $-\mathrm{A}$ is the infinitesimal generator of an analytic semigroup on a Banach space X, $k(\cdot)$ is an operator-valued map on $[0, +\infty)$ and u_0 belongs to X. The above problem arises e.g. when studying the Liapunof stability character of the equilibrium solutions for Volterra's integro-partial differential equation [5].

In the finite-dimensional case $\mathrm{X} = \mathrm{R}^n$ the condition $\det[\zeta \mathrm{I} + \mathrm{A} - k^*(\zeta)] \neq 0$ for $\mathrm{Re}\, \zeta \geq 0$ is known to ensure, if $k(\cdot) \in \mathrm{L}^1$, the asymptotic stability of the trivial solution with respect to (P) [2]. Under suitable separation assumptions on the spectrum of the operator A, we shall generalize the above result to the present situation (for this purpose, simpler techniques can be used in the case of a scalar-valued kernel $k(\cdot)$ [6]); use will be made of a projection procedure onto the invariant subspaces of X associated with the operator A, which gives rise to a family of approximating problems.

2. STATEMENT OF THE RESULTS

Let X denote a complex Banach space, endowed with the norm $\mathrm{X} \ni u \to |u|_{\mathrm{X}}$. We shall denote by $\mathscr{L}(\mathrm{X})$ the Banach algebra of linear bounded operators on X endowed with the operatorial norm $\mathscr{L}(\mathrm{X}) \ni \mathrm{B} \to \|\mathrm{B}\|$. By $\mathrm{L}^p(0, +\infty; \mathrm{Y})$ we mean the Banach space of measurable functions from

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$[0, +\infty)$ to the Banach space Y (norm $|\cdot|_Y$), such that $\int_0^{+\infty} |f(t)|_Y^p \, dt < +\infty$, endowed with the usual norm $(1 \le p < +\infty)$; we shall also be dealing with the Fréchet space $L^1_{loc}(0, +\infty; Y)$ endowed with the family of seminorms $f \to p_T(f) := \int_0^T |f(t)|_Y \, dt \; (T > 0)$. For any $k(\cdot) \in : L^1(0, +\infty; Y)$, the notation $k^*(\cdot)$ for the Laplace transform of $k(\cdot)$ will be adopted.

The problem (P) will be studied under the following assumptions:

A_1) $-A$ is the infinitesimal generator of an analytic semigroup $T(\cdot)$ on X, such that $\| T(t) \| \le M \exp(\omega t) \; (t \ge 0 ; M \ge 1 ; \omega \in R)$;

A_2) $(\zeta + A)^{-1}$ is compact for some ζ in the resolvent set $\rho(-A)$; moreover, there exists a sequence $\{P_n\} \subseteq \mathscr{L}(X)$ of projections such that $(n \in N)$:

(i) $\dim P_n < \infty$, $P_n P_{n+1} = P_{n+1} P_n = P_n$;

(ii) $P_n A \subseteq AP_n$;

(iii) there exist $\vartheta_0 \in [0, 1)$ and $\zeta_0 \in \rho(-A)$ such that $(I - P_n)(\zeta_0 + A)^{-\vartheta_0}$ converges in the strong sense as n diverges;

(iv) there exists $\bar{n} \in N$ such that, for any $n \ge \bar{n}$, the type ω_n of the analytic semigroup $(I - P_n) T(\cdot)$ is strictly negative.

Further we have the following assumptions:

k_1) $k(\cdot) \in L^1(0, +\infty; \mathscr{L}(X))$;

k_2) there exists $\alpha > 2\vartheta_0 - 1$ such that $\| (tk)^*(\zeta) \| \le \text{const.} (1 + |\zeta|)^{-\alpha}$ for any $\zeta \in C_+ := \{\zeta \in C \mid Re\, \zeta \ge 0\}$ (where $(tk)(t) := tk(t), t \ge 0$);

H) for any $\zeta \in C_+$, there exists $D(\zeta) := [\zeta I + A - k^*(\zeta)]^{-1} \in \mathscr{L}(X)$.

As a consequence of the assumption A_2), we can think of the Banach space X as a direct sum: $X = X_n \oplus \tilde{X}_n$, where $X_n := P_n X \; (\dim X_n < \infty)$ and $\tilde{X}_n := (I - P_n) X$; moreover, $X_n \subseteq X_{n+1}$ and $\tilde{X}_n \supseteq \tilde{X}_{n+1} \; (n \in N)$. The assumption A_2), (iii) is obviously satisfied with $\vartheta_0 = 0$ if X is Hilbert and A a normal operator; it can also be proved to hold in the case A is a second-order formally self-adjoint elliptic operator on $X = C(\overline{\Omega})$, $C(\overline{\Omega})$ denoting the Banach space of continuous functions (endowed with the supremum norm) on some bounded regular domain $\overline{\Omega} \subset R^d$, $d \le 3$ [5].

The regularity condition k_2) is satisfied e.g. if $(tk(\cdot)) \in L^1$ whenever $\vartheta_0 < 1/2$. Eventually, the part $-(I - P_n) A$ of $-A$ in $\tilde{X}_n$ is easily seen to generate an analytic semigroup on $\tilde{X}_n$, so that the requirement A_2), (iv) makes sense.

By a fundamental solution of problem (P) we mean a map $\mathscr{S} : [0, +\infty) \to \mathscr{L}(X)$ such that

$$\mathscr{S}(t) = T(t) + [T * k * \mathscr{S}](t) \qquad (t \geq 0), \tag{1}$$

where the shorthand $(f * g)(t) := \int_0^t ds\, f(t-s)\, g(s)\ (t \geq 0)$ for the convolution is used. As is well known, if a fundamental solution of (P) exists, $\mathscr{S}(\cdot)\, u_0$ is a mild solution of (P) [4]; moreover, due to assumption A_1), the map $t \to \mathscr{S}(t)\, u_0$ is continuously differentiable for any $t > 0$ and satisfies (P).

The trivial solution $u = 0$ is said to be (Liapunof) stable with respect to (P) if for any $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that $|u_0|_X < \delta$ implies $|u(t)|_X < \varepsilon$ for any $t \geq 0$; it is said to be asymptotically stable if in addition it is attractive, namely if there exists $\eta > 0$ such that $|u_0|_X < \eta$ implies $|u(t)|_X \to 0$ as $t \to +\infty$.

Due to assumptions A_1) and k_1), the existence of a unique fundamental solution $\mathscr{S}(\cdot) \in L^1_{loc}(0, +\infty; \mathscr{L}(X))$ follows from a standard contraction mapping argument and from the a priori estimate:

$$\|\mathscr{S}(t)\| \leq M \exp[\omega + M\, |k|_{L^1}]\, t \qquad (t \geq 0),$$

which also proves the existence of the Laplace transform $\mathscr{S}^*(\cdot)$ of $\mathscr{S}(\cdot)$ in the complex half-plane $\mathrm{Re}\, \zeta > \omega + M\, |k|_{L^1}$; then it is easily seen that $\mathscr{S}^*(\zeta) = D(\zeta)$ whenever both quantities make sense. Conversely, under the above assumptions suitable boundedness and decrease properties for the fundamental solution can be derived, which imply the following main result.

THEOREM 1. *Let the assumption* A_1) — H) *be satisfied. Then the trivial solution is asymptotically stable with respect to problem* (P).

The proof will follow (see Section 5) from some preliminary consequences of the assumptions (see Section 3) and from the properties of the solutions of a family of problems, which approximate (P) in a suitable sense (Section 4).

3. PRELIMINARY RESULTS

As a first consequence of assumptions A_1), A_2) we have the following lemma.

LEMMA 1. *Let* A_1), A_2) *be satisfied. Then the following hold:*

(i) $\|(I - P_n)\, T(t)\| \underset{n\to\infty}{\to} 0$ *for any* $t > 0$;

(ii) $\|(I - P_n)(\zeta + A)^{-\vartheta}\| \underset{n\to\infty}{\to} 0$ *for any* $\vartheta > \vartheta_0$ *and* $\zeta \in \rho(-A)$; *the convergence is uniform in any closed subset of* C_+ *whose intersection with the spectrum* $\sigma(-A)$ *is empty.*

Proof. Let $\{B_n\} \subseteq \mathscr{L}(X)$, $B_n \to 0$ in the strong sense and $C \in \mathscr{L}(X)$ be compact; then it is easily proved that $\|B_n C\| \to 0$ as n diverges. Now the proof of (i) follows from the identity

$$(I - P_n) T(t) = (I - P_n)(\zeta_0 + A)^{-\vartheta_0} (\zeta_0 + A)^{\vartheta_0} T(t) \qquad (t \geq 0; \zeta_0 \in \rho(-A))$$

and the assumed regularization properties of $T(t)$ for any $t > 0$. The proof of (ii) is similar.

PROPOSITION 1. *Let* A_1), A_2) *be satisfied. Then*

$$(2) \qquad \int_0^{+\infty} \|(I - P_n) T(t)\|^\beta \, dt \underset{n\to\infty}{\longrightarrow} 0 \qquad \textit{for any} \quad \beta \in [1, 1/\vartheta_0).$$

Proof. Due to Lemma 1 and the dominated convergence theorem, suffice it to exhibit a map $\varphi(\cdot) \in L^\beta(0, +\infty)$ such that $\|(I - P_n) T(t)\| \leq \varphi(t)$ for almost every $t > 0$. In fact, according to hypotheses A_2), (i)—(iii), there exist $\bar{n} \in N$ and a constant $B > 0$ such that:

$$\|(I - P_n) T(t)\| = \sup_{x \in X, |x|_X \leq 1} |(I - P_n) T(t) x|_X \leq \sup_{y \in \tilde{X}_n, |y|_X \leq B} |(\zeta_0 + A)^{\vartheta_0} T(t) y|_X \leq$$

$$\leq \sup_{y \in \tilde{X}_{\bar{n}}, |y|_X \leq B} |(\zeta_0 + A)^{\vartheta_0} T(t) y|_X \leq B M_{\bar{n}} \exp(\omega_{\bar{n}} t) t^{-\vartheta_0} = : \varphi(t) \qquad (t > 0)$$

with suitable constant $M_{\bar{n}} > 0$; then the conclusion follows.

PROPOSITION 2. *Let* A_1), k_1) *and* H) *be satisfied. Then there exists* $h > 0$ *such that the following inequality holds*:

$$(3) \qquad \|D(\zeta)\| \leq h/(1 + |\zeta|) \qquad (\mathrm{Re}\,\zeta \geq 0).$$

Proof. According to assumption A_1), there exist $\gamma \in R$ and $\vartheta \in (0, \pi/2)$ such that $\rho(-A) \supseteq \{\zeta \in C \mid |\arg(\zeta - \gamma)| \leq \pi/2 + \vartheta\}$. Let us limit ourselves to the case $\gamma > 0$ (the proof is easier if $\gamma \leq 0$). Define $V := \{\zeta \in C_+ \mid |\arg(\zeta - \gamma)| \geq \pi/2 + \vartheta\}$ and $S_\delta := \{\zeta \in C_+ \mid |\zeta| \leq \delta\} (\delta > 0)$; let moreover $\tilde{V}$ (resp. $\tilde{S}_\delta$) denote the complement of V (resp. S_δ) with respect to the closed half-plane C_+.

Due to assumption A_1), there exist $\delta > 0$, $c_1 = c_1(\delta) > 0$ such that:

$$\|(\zeta + A)^{-1}\| \leq c_1/(1 + |\zeta|) \qquad (\zeta \in \tilde{V} \cap \tilde{S}_\delta).$$

By assumption H) we get the estimate:

$$\|D(\zeta)\| \leq 2\, c_1/(1 + |\zeta|) \qquad (\zeta \in \tilde{V} \cap \tilde{S}_{\bar{\delta}}),$$

where $\bar{\delta} := \max\{\delta, 2|k|_{L_1} c_1 - 1\}$. Due to the analiticity of the map $D(\cdot)$, in the compact subset $V \cup S_{\bar{\delta}}$ we have the similar estimate:

$$\| D(\zeta) \| \leq c_2/(1 + |\zeta|) \qquad (\zeta \in V \cup S_{\bar{\delta}})$$

for a suitable constant $c_2 > 0$; then the result follows with $h := \max\{2c_1, c_2\}$.

In the finite-dimensional case, the inequality (3) implies that

$$\mathscr{S}(\cdot) \in L^2(0, +\infty; \mathscr{L}(X)) \quad [3, \text{Ch. } 1].$$

4. APPROXIMATE PROBLEMS

Let us consider the family of approximate problems:

$$(4_n) \qquad \mathscr{S}_n(t) = T(t) P_n + [T * (P_n k P_n) * \mathscr{S}_n](t) \qquad (n \in N; t \geq 0).$$

The same arguments used in connection with problem (1) prove the existence of a unique solution $\mathscr{S}_n(\cdot) \in L^1_{loc}(0, +\infty; \mathscr{L}(X))$ of (4_n), for any $n \in N$; moreover, $\mathscr{S}_n(\cdot)$ is continuous in the uniform sense as $t \to 0^+$. By the uniqueness of the solution of (4_n) we also have $\mathscr{S}_n(t) P_n = P_n \mathscr{S}_n(t) = \mathscr{S}_n(t)$.

In connection with problems (4_n) the following proposition is of use.

PROPOSITION 3. *Let* $A_1, A_2), k_1)$ and H) *be satisfied. Then there exists* $n_0 \in N$ *such that* $[\zeta I + A - P_n k^*(\zeta)]^{-1} \in \mathscr{L}(X)$ *for any* $n > n_0$ $(n \in N)$ *and* $\zeta \in C_+$.

Proof. Suppose false; then there exist sequences $\{\zeta_k\} \subseteq C_+$, $\{u_k\} \subseteq X$, $|u_k|_X = 1$ such that

$$\zeta_k u_k + A u_k - P_{n_k} k^*(\zeta_k) u_k = 0 \qquad (k \in N).$$

We may suppose either $\zeta_k \to \hat{\zeta}$ or $|\zeta_k| \to +\infty$ as $k \to \infty$. If $|\zeta_k| \to +\infty$, due to Lemma 1, (ii) and the above equation the contradiction $|u_k|_X \to 0$ easily follows; thus we can assume $\zeta_k \to \hat{\zeta}$ ($\operatorname{Re} \hat{\zeta} \geq 0$). Due to the equality

$$u_k = (\zeta_0 - \zeta_k)(\zeta_0 + A)^{-1} u_k - (\zeta_0 + A)^{-\vartheta_0} P_{n_k} (\zeta_0 + A)^{-1+\vartheta_0} k^*(\zeta_k) u_k,$$

by a compactness argument, we may assume $\hat{u} \in X$, $|\hat{u}|_X = 1$ to exist, such that $|u_k - \hat{u}|_X \to 0$ as k diverges. As a consequence, $\hat{u}$ is in the domain of $(\zeta_0 + A)^{1-\vartheta_0}$ and the following equality holds:

$$(\zeta_0 + A)^{1-\vartheta_0} \hat{u} = (\zeta_0 - \hat{\zeta})(\zeta_0 + A)^{-\vartheta_0} \hat{u} + (\zeta_0 + A)^{-\vartheta_0} k^*(\hat{\zeta}) \hat{u};$$

this in turn implies $\hat{u} \in D(A)$ and the relation $[\hat{\zeta} I + A - k^*(\hat{\zeta})] \hat{u} = \mathbf{0}$, whence the conclusion follows.

COROLLARY 1. *Let* $A_1), A_2) k_1)$ *and* H) *be satisfied. Then there exists* $n_0 \in N$ *such that* $D_n(\zeta) := [\zeta I + A - P_n k^*(\zeta)]^{-1} P_n \in \mathscr{L}(X_n)$ *for any* $n > n_0$ *and* $\zeta \in C_+$.

Proof. Follows from Proposition 3, as the operator $\zeta \mathrm{I} + \mathrm{A} - \mathrm{P}_n k^*(\zeta)$ leaves the subspace X_n invariant.

COROLLARY 2. *Let* $\mathrm{A}_1)$, $\mathrm{A}_2)$, $k_1)$ *and* H) *be satisfied. Then* $\mathcal{S}_n(\cdot) \in \mathrm{L}^1(0, +\infty; \mathcal{L}(\mathrm{X}))$ *for any* $n > n_0$.

Proof. Follows from Corollary 1 by finite-dimensional results [2].

A property of uniform boundedness of the sequences $\{\mathrm{D}_n(\cdot)\}$, $\{(\mathrm{dD}_n/\mathrm{d}\zeta)(\cdot)\}$ in C_+ is the content of the following propositions.

PROPOSITION 4. *Let* $\mathrm{A}_1)$, $\mathrm{A}_2)$, $k_1)$ *and* H) *be satisfied. Then there exist* $n_1 \in \mathrm{N}$, $n_1 > n_0$ *and* $\tilde{h} > 0$ *such that the following inequality holds*:

$$\|\mathrm{D}_n(\zeta)\| \leq \tilde{h}/(1 + |\zeta|)^{1-\vartheta_0} \qquad (n > n_1, \mathrm{R}e\,\zeta \geq 0). \tag{5}$$

Proof. (i) As a consequence of Lemma 1, there exist $\eta > 0$ and $n' \in \mathrm{N}$ such that

$$\sup_{\zeta \in \tilde{\mathrm{S}}_\eta} \|(\mathrm{I} - \mathrm{P}_n)\,\mathrm{D}(\zeta)\| \leq 1/(2\,|k|_{\mathrm{L}^1}) \qquad (n > n').$$

Then, if $n > n^* := \max\{n_0, n'\}$ and $\zeta \in \tilde{\mathrm{S}}_\eta$, the following relation is easily seen to hold:

$$\|\mathrm{D}_n(\zeta)\| \leq h_1 \|\mathrm{P}_n \mathrm{D}(\zeta)\| = h_1 \|\mathrm{P}_n(\zeta + \mathrm{A})^{-1}\{\mathrm{I} - k^*(\zeta)(\zeta + \mathrm{A})^{-1}\}^{-1}\| \leq$$
$$\leq h_2(1 + |\zeta|)^{\vartheta_0 - 1} \qquad (h_1, h_2 > 0).$$

(ii) Now we claim that $n^{**} \geq n_0$, $n^{**} \in \mathrm{N}$ exists, such that for any integer $n > n^{**}$ $\sup_{\zeta \in \mathrm{S}_\eta} \|\mathrm{D}_n(\zeta)\| \leq$ const. Suppose false; then there exist sequences $\{\zeta_k\} \subset \mathrm{S}_\eta$, $\zeta_k \to \tilde{\zeta}$ as k diverges, $\{u_k\} \subset \mathrm{X}$, $|u_k|_{\mathrm{X}} = 1$, such that the sequence $\{v_k\} := \{\mathrm{D}_{n_k}(\zeta_k)\, u_k\} \subset \mathrm{X}$ diverges as k diverges. Set $w_k := v_k/|v_k|_{\mathrm{X}}$; from the equality

$$(\zeta_0 + \mathrm{A})^{-1}(\zeta_0 - \zeta_k)\, w_k + \mathrm{P}_{n_k}(\zeta_0 + \mathrm{A})^{-1}(|v_k|_{\mathrm{X}})^{-1} u_k =$$
$$= w_k - \mathrm{P}_{n_k}(\zeta_0 + \mathrm{A})^{-1} k^*(\zeta_k)\, w_k$$

we may assume $\tilde{w} \in \mathrm{X}$ to exist, such that $|w_k - \tilde{w}|_{\mathrm{X}} \to 0$ as $k \to \infty$. It follows that $\tilde{w} \in \mathrm{D}(\mathrm{A})$ and $[\tilde{\zeta}\mathrm{I} + \mathrm{A} - k^*(\tilde{\zeta})]\,\tilde{w} = 0$, so that by contradiction the claim is proved. Now the conclusion follows with $n_1 := \max\{n^*, n^{**}\}$ and a suitable constant $h > 0$.

PROPOSITION 5. *Let* $\mathrm{A}_1)$, $\mathrm{A}_2)$, $k_1)$, $k_2)$ *and* H) *be satisfied. Then there exists* $\bar{h} > 0$ *such that the following inequality holds*:

$$\|(\mathrm{dD}_n/\mathrm{d}\zeta)(\zeta)\| \leq \bar{h}/(1 + |\zeta|)^{\nu} \qquad (n > n_1;\ \mathrm{R}e\,\zeta \geq 0), \tag{6}$$

where $\nu := \min\{2 - \vartheta_0, 2(1 - \vartheta_0) + \alpha\} > 1$.

Proof. Follows from assumption k_2) and Proposition 4, due to the identity:

$$(\mathrm{dD}_n/\mathrm{d}\zeta)(\zeta) = -[\zeta \mathrm{I} + \mathrm{A} - \mathrm{P}_n k^*(\zeta)]^{-1} [\mathrm{I} - \mathrm{P}_n (\mathrm{d}k^*/\mathrm{d}\zeta)(\zeta)] \mathrm{D}_n(\zeta) \qquad (n > n_0 ; \mathrm{Re}\, \zeta \geq 0).$$

5. CONVERGENCE RESULTS

It is convenient to consider the projected equalities:

$$(7_n) \qquad \mathrm{P}_n \mathscr{S}(t) = \mathrm{T}(t)\mathrm{P}_n + [(\mathrm{TP}_n) * k * \mathscr{S}](t) \qquad (t \geq 0).$$

By standard uniqueness results and the very definition of $\mathscr{S}_n(\cdot)$, the following relation is proved to hold:

$$(8_n) \qquad \mathscr{S}_n(t) = \mathrm{P}_n \mathscr{S}(t) - [\mathscr{S}_n * k * (\mathrm{I} - \mathrm{P}_n)\mathscr{S}](t) \qquad (n \in \mathrm{N} ; t \geq 0).$$

PROPOSITION 6. *Let* A_1) *and* A_2) *be satisfied. Then the sequence* $\{(\mathrm{I} - \mathrm{P}_n)\mathscr{S}(\cdot)\}$ *is infinitesimal in* $\mathrm{L}^1_{\mathrm{loc}}(0, +\infty ; \mathscr{L}(\mathrm{X}))$.

Proof. Due to Proposition 1 and the boundedness of $\mathscr{S}(\cdot)$ on the bounded subsets of $[0, +\infty)$, we have

$$\sup_{t \in [0, \bar{t}]} \| (\mathrm{I} - \mathrm{P}_n)\mathrm{T} * k * \mathscr{S} \| (t) \underset{n \to \infty}{\longrightarrow} 0 \qquad \text{for any} \quad \bar{t} > 0;$$

then the conclusion follows from (7_n) according to Proposition 1.

Concerning the sequence of approximate solutions $\{\mathscr{S}_n(\cdot)\}$, we have the following result.

PROPOSITION 7. *Let* A_1) — H) *be satisfied. Then there exists an integer* n_1 *such that, for any* $n > n_1$ $(n \in \mathrm{N})$, *the following hold*:

(i) $\| t\mathscr{S}_n(t) \| \leq m_1$ *for any* $t > 0$ $(m_1 > 0)$;

(ii) *the sequence* $\{\mathscr{S}_n(\cdot)\}$ *is bounded in* $\mathrm{L}^1_{\mathrm{loc}}(0, +\infty ; \mathscr{L}(\mathrm{X}))$.

Proof. According to Corollary 2 and Proposition 5, for any integer n the map $\zeta \to (\mathrm{dD}_n/\mathrm{d}\zeta)(\zeta)$ is the Laplace transform of $-(t\mathscr{S}_n)(\cdot)$; then the conclusion follows from the inversion formula of the Laplace transform [1] due to the uniform estimate (6). This proves (i); as for (ii), suffice it to prove that $\int_0^\tau \| \mathscr{S}_n(t) \| \mathrm{d}t \leq$ const. for a suitable $\tau > 0$. According to Proposition 1, $\int_0^{+\infty} \| (\mathrm{I} - \mathrm{P}_n)\mathrm{T} * k \| (t)\, \mathrm{d}t \to 0$ as $n \to \infty$; thus we can choose $\tau > 0$ such that $\int_0^\tau \| (\mathrm{P}_n \mathrm{T}) * k \| (t)\, \mathrm{d}t \leq 1/2$; due to (4_n) and Proposition 1 the result follows.

Remark. By the above argument the sequence $\{\mathscr{S}_n(\cdot)\}$ can be proved to be bounded in $L^\beta(0, +\infty; \mathscr{L}(X))$ for any $\beta \in (1, 1/\vartheta_0)$.

PROPOSITION 8. *Let* $A_1) - H)$ *be satisfied. Then the sequence* $\{(\mathscr{S}_n - P_n\mathscr{S})(\cdot)\}$ *is infinitesimal in* $L^1_{loc}(0, +\infty; \mathscr{L}(X))$.

Proof. Follows from the identity (8_n) according to Propositions 6 and 7, (ii) above.

Finally, Theorem 1 is an easy consequence of the following result.

PROPOSITION 9. *Let* $A_1) - H)$ *be satisfied. Then the following inequality holds*:

$$\|\mathscr{S}(t)\| \leq \text{const}/(1+t)^{-1} \tag{9}$$

Proof. Due to Propositions 6 and 8, a subsequence $\{\mathscr{S}_{n_k}(\cdot)\}$ exists, such that $\|\mathscr{S}_{n_k}(t) - \mathscr{S}(t)\| \to 0$ as $k \to \infty$ for almost every $t \geq 0$. By Proposition 7, (i) and the boundedness of $\mathscr{S}(\cdot)$ on the bounded subsets of $[0, +\infty)$ we get the inequality (9).

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