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On non-uniform partial stability and perturbations for delay systems

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Equazioni differenziali ordinarie. — *On non-uniform partial stability and perturbations for delay systems.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si studiano effetti delle perturbazioni sulla stabilità parziale per insiemi antinvarianti asintoticamente.

§ 1. INTRODUCTION

In [1], we obtained a characterization of the non-uniform partial asymptotic exponential stability of an invariant set in terms of Lyapunov functionals. In addition we constructed smooth Lyapunov functionals for the generalized partial asymptotic equi-exponential stability of an invariant set relative to a dealy system.

In this paper we investigate the behaviour of perturbed non-linear differential systems with delay whenever the unperturbed system has non-uniform partial asymptotic exponential stability property for some kind of invariant sets. We also use our result in [1] to find conditions for the generalized partial asymptotic equi-exponential stability property to be preserved under certain perturbations. Our results generalize some results of [2] and complements [1].

§ 2. NOTATIONS AND DEFINITIONS

We shall consider the functional differential system

$$(1) \qquad \begin{cases} \dot{v}(t) = f(t, v_t, w_t) \\ \dot{w}(t) = g(t), v_t, w_t) \end{cases} \qquad v_{t_0} = \phi_0 , \quad w_{t_0} = \psi_0$$

where $t \in R^+ = [0, \infty)$, $v \in R^n$, $w \in R^m$ and f, g are continuous functions from $R^+ \times C([-h, 0], R^n \times C([-h, 0], R^m)$ into R^n and R^m respectively, $C([-h, 0], R^n)$ being the space of continuous functions from $[-h, 0]$ into R^n. Here for $h > 0$, $l^n = C([-h, 0], R^n)$ and $\|\phi\|_0 = \max_{-h \le s \le 0} \|\phi(s)\|$, $\|\cdot\|$ being any convenient norm in R^n. If $(t_0, \phi, \psi) \in R^+ \times l^n \times l^m$, we denote by $v_t(t_0; \phi, \psi)$ and $w_t(t_0; \phi, \psi)$ the solution of (1) such that $v_{t_0} = \phi$ and $w_{t_0} = \psi$. For $t \ge t_0$, $v_t(t_0; \phi, \psi) \in l^n$ and $w_t(t_0; \phi, \psi) \in l^m$.

(*) Nella seduta del 21 aprile 1979.

Along with (1) we consider the perturbed system

$$
\begin{cases} \dot{y}(t) = f(t, y_t, z_t) + G(t, y_t, z_t) \\ \dot{z}(t) = g(t, y_t, z_t) + F(t, y_t, z_t) \end{cases} \tag{2}
$$

where $G(t, \phi, \psi)$ and $F(t, \phi, \psi)$ are continuous mappings from $R^+ \times C([-h, 0], R^n) \times C([-h, 0], R^m)$ into R^n and R^m respectively. We assume in addition that $\|G(t, 0, 0)\|_0 + \|F(t, 0, 0)\|_0 \equiv 0$.

We denote by $y_t(t_0; \phi, \psi)$ and $z_t(t_0; \phi, \psi)$ a solution of (2) such that $y_{t_0} = \phi$ and $z_{t_0} = \psi$.

For the definitions of ϕ-asymptotic self-invariance (ϕ-ASI), ψ-asymptotic self-invariance (ψ-ASI) and asymptotic self-invariance of the set $\phi = 0$, $\psi = 0$ with respect to the system (1) see [1, § 2]. Moreover the partial equi-stability and the partial equi-exponential asymptotic stability properties of the ϕ-ASI set $\phi = 0$, $\psi = 0$ with respect to the system (1) are defined as in [1] as well. The generalized partial equi-exponential asymptotic stability property of the ϕ-ASI set $\phi = 0$ and $\psi = 0$ with respect to (1) is defined in [1].

DEFINITION 2.1. The ASI set $u = 0$ of the scalar differential equation

$$
\frac{du}{dt} = g(t, u), \; u(t_0) = u_0 > 0 \tag{3}
$$

where $g \in C(R^+ \times R^+, R^+)$ is said to be equistable if for each $t_0 \in R^+$, there exists $M(t_0, \tau) > 0$ and $q \in \mathscr{L}$ where $\mathscr{L}$ is as defined in [1], such that

$$
u(t; t_0, u_0) \leq M(t_0, \tau) u_0 + q(t_0); \; t \geq t_0 .
$$

§ 3. PERTURBATION RESULTS

We shall now use our results in [1] to investigate the behaviour of the ϕ-ASI set $\phi = 0$, $\psi = 0$ under constantly acting perturbations. We give sufficient conditions for the ϕ-ASI set $\phi = 0$, $\psi = 0$ to be partially equi-stable, or partially equi-exponential asymptotic stable with respect to the system (2) when the system (1) possesses the partial equi-exponential asymptotic stability property. We also give conditions for the preservation of generalized partial equi–exponential asymptotic stability of the ϕ-ASI set $\phi = 0$, $\psi = 0$ under such perturbations.

THEOREM 3.1. *Assume that*

(i) *the* ϕ-ASI set $\phi = 0$, $\psi = 0$ *of the system* (1) *is partially equi-exponential asymptotically stable and any two solutions*

$$
(v_t(t_0; \phi_1, \psi_1), w_t(t_0; \phi_1, \psi_1))
$$

and

$$
(v_t(t_0; \phi_2, \psi_2), w_t(t_0; \phi_2, \psi_2))
$$

of (1) *satisfy*

$$\| v_t(t_0; \phi_1, \psi_1) - v_t(t_0; \phi_2, \psi_2 \| \leq K(t_0, \tau)(\| \phi_1 - \phi_2 \|_0 + \| \psi_1 - \psi_2 \|_0) e^{-\alpha(t-t_0)}$$

(ii) *the system* (2) *admits a unique solution and for* $t \geq t_0$ $(t, \phi, \psi) \in R^+ \times C_\rho \times C_\beta$,

$$\| G(t, \phi, \psi) \|_0 + \| F(t, \phi, \psi) \|_0 \leq \omega(t, \| \phi \|_0)$$

where $\omega \in C(R^+ \times R^+, R^+)$ *and is non-decreasing in* u *for each* $t \in R^+$.

(iii) *the function* $H(t, t_0)$ *is partially differentiable with respect to* t_0 *and*

$$\operatorname*{Sup}_{\delta \geq 0} \left\{ -\frac{\partial H}{\partial t_0}(t+\delta, t) e^{\alpha\delta} \right\} \leq \eta(t)$$

where $\eta \in C(R^+, R^+)$ *and* $\int_t^{t+1} \eta(s)\, ds \to 0$ *as* $t \to \infty$.

Then the equi-stability of the ASI set $u = 0$ of the scalar differential equation

$$(4) \qquad \frac{du}{dt} = -\alpha u + K(t, \tau)\, \omega(t, u) + \eta(t), \; u(t_0) = u_0 > 0$$

implies the partial equi-stability of the ϕ-ASI set $\phi = 0$, $\psi = 0$ of the perturbed system (2).

Proof. Assume that $u = 0$ is ASI with respect to (4) and equistable, then

$$u(t, t_0, u_0) \leq M(t_0, \tau) u_0 + q(t_0), \qquad t \geq t_0,$$

where $u_0 > 0$, $q \in \mathscr{L}$ and $M \in C(R^+ \times R^+, R^+)$. Let $0 < \varepsilon \leq \rho$, then there exists $T(\varepsilon) > 0$ such that $q(t_0) < \frac{\varepsilon}{2}$, if $t_0 \geq T(\varepsilon)$. For $t_0 \geq T(\varepsilon)$, choose $u_0 < \frac{\varepsilon}{2 M(t_0, \tau)}$, then

$$u(t; t_0, u_0) < \varepsilon, \qquad \text{for } t \geq t_0 \geq T(\varepsilon)$$

provided $u_0 < \frac{\varepsilon}{2 M(t_0, \tau)}$.

Let $y_t(t_0; \phi_0, \psi_0)$, $z_t(t_0; \phi_0, \psi_0)$ be any solution of the perturbed system (2) such that $\| \phi_0 \|_0 + \| \psi_0 \|_0 < \frac{\varepsilon}{2 K(t_0, \tau) M(t_0, \tau)}$ and for $t \geq T(\varepsilon)$. Setting $\phi = y_t(t_0; \phi_0, \psi_0)$ and $\psi = z_t(t_0; \phi_0, \psi_0)$ we have by uniqueness of solutions $y_{t+h}(t_0; \phi_0, \psi_0) = y_{t+h}(t_0; \phi, \psi)$ $h \geq 0$ and $z_{t+h}(t_0; \phi_0, \psi_0) = z_{t+h}(t_0; \phi, \psi)$.

Let $v_{t+h}(t;\phi,\psi), w_{t+h}(t;\phi,\psi)\,(h \geq 0)$ be a solution of (1) through (t,ϕ,ψ), then provided $\| v_t(t_0;\phi_0,\psi_0)\| < \rho$ and $\| w_t(t_0;\phi_0,\psi_0)\|_0 < \beta$ for $t \geq t_0 \geq T(\varepsilon)$, by the Theorem 2.7 of [1] there exists a Lyapunov functional $W(t,\phi,\psi)$ such that

$$
\begin{aligned}
&D^+ W(t,\phi,\psi)_{(2)} \\
&\quad = \lim_{h\to 0^+} \operatorname{Sup} \frac{1}{h} [W(t+h, y_{t+h}(t;\phi,\psi), z_{t+h}(t;\phi,\psi)) \\
&\qquad - W(t+h, v_{t+h}(t;\phi,\psi), w_{t+h}(t;\phi,\psi)) \\
&\qquad + W(t+h, v_{t+h}(t;\phi,\psi), w_{t+h}(t;\phi,\psi)) - W(t,\phi,\psi)] \\
&\quad \leq D^+ W(t,\phi,\psi)_{(1)} + \lim_{h\to 0^+} \operatorname{Sup} \frac{1}{h} [K(t+h,\tau)\{\| y_{t+h} - v_{t+h}\|_0 + \\
&\qquad + \| z_{t+h} - w_{t+h}\|_0\}] \\
&\quad \leq D^+ W(t,\phi,\psi)_{(1)} + K(t,\tau)\{\| y'(t;\phi,\psi)(t) - v'(t;\phi,\psi)(t)\| + \\
&\qquad + \| z'(t;\phi,\psi)(t) - w'(t;\phi,\psi)(t)\|\} \\
&\quad \leq -\alpha W(t,\phi,\psi) + \eta(t) + K(t,\tau)[\| G(t,\phi,\psi)\|_0 + \| F(t,\phi,\psi)\|_0] \\
&\quad \leq -\alpha W(t,\phi,\psi) + \eta(t) + K(t,\tau)\, w(t, \|\phi\|_0)
\end{aligned}
$$

where $D^+ W(t,\phi,\psi)_{(2)}$ and $D^+ W(t,\phi,\psi)_{(1)}$ denote the Dini derivatives with respect to the systems (2) and (1) respectively.

Since $\phi = y_t(t;\phi,\psi)$ and $\psi = z_t(t;\phi,\psi)$ assumption (ii) and conclution (II) of Theorem 2.7 of [1] imply

$$
\begin{aligned}
D^+ W(t, y_t(t;\phi,\psi), z_t(t;\phi,\psi)) &\leq -\alpha W(t, y_t, z_t) + \\
&\quad + \eta(t) + K(t,\tau)\,\omega(t, W(t, y_t, z_t)).
\end{aligned}
$$

Hence Theorem 1.4.1 of [3] imply

$$
W(t, y_t(t_0;\phi_0,\psi_0), z_t(t_0;\phi_0,\psi_0)) \leq r(t,t_0,u_0) \qquad t \geq t_0 \geq T(\varepsilon)
$$

where $r(t,t_0,u_0)$ is the maximal solution of (4) such that

$$
u_0 = K(t_0,\tau)(\|\phi_0\|_0 + \|\psi_0\|_0).
$$

By Theorem 2.7 of [1] we have for $t \geq t_0 \geq T(\varepsilon)$

$$
\| y_t(t_0;\phi_0,\psi_0)\|_0 \leq W(t, y_t(t_0;\phi_0,\psi_0), z_t(t_0;\phi_0,\psi_0)) \leq r(t,t_0,u_0).
$$

However,

$$
u_0 = K(t_0,\tau)(\|\phi_0\|_0 + \|\psi_0\|_0) \quad \text{and} \quad \|\phi_0\|_0 + \|\psi_0\|_0 \leq \frac{\varepsilon}{2\,K(t_0,\tau)\,M(t_0,\tau)}
$$

with $t_0 \geq T(\varepsilon)$, hence $u_0 < \dfrac{\varepsilon}{2 M(t_0, \tau)}$. The assumption on the solutions of (4) implies

$$r(t, t_0, u_0) < \varepsilon \qquad \text{for} \qquad t \geq t_0 \geq T(\varepsilon).$$

Hence

$$\| y_t(t_0; \phi_0, \psi_0) \| < \varepsilon \qquad \text{for} \qquad t \geq t_0 \geq T(\varepsilon)$$

provided $\| \phi_0 \|_0 + \| \psi_0 \|_0 < \dfrac{\varepsilon}{2 K(t_0, \tau) M(t_0, \tau)}$. Finally by assumption on solutions of (4)

$$\| y_t(t_0; \phi_0, \psi_0) \|_0 \leq r(t, t_0, u_0) \leq M(t_0, \tau) K(t_0, \tau) (\| \phi_0 \|_0 + \| \psi_0 \|_0) + q(t_0), \quad t \geq t_0.$$

Moreover,

$$\| y_t(t_0; 0, 0) \|_0 \leq q(t_0), \qquad t \geq t_0$$

so that the set $\phi = 0$, $\psi = 0$ is ϕ-ASI with respect to the perturbed system (2) [cf. 1] and is partially equistable or equistable with respect to the y-component.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. Then the ASI set $u = 0$ of (4) is equi-exponentially asymptotically stable implies that the ϕ-ASI set $\phi = 0$, $\psi = 0$ of the perturbed system (2) is partially equi-exponential asymptotically stable.*

Proof. By hypothesis $u = 0$ of (4) is equi-exponential asymptotically stable, hence for $u_0 > 0$ and $u(t, t_0, u_0)$ any solution of (4)

$$u(t, t_0, u_0) \leq M(t_0, \tau) u_0 e^{-\alpha(t-t_0)} + H(t, t_0), \qquad t \geq t_0$$

where $\alpha > 0$, $H \in C(R^+ \times R^+, R^+)$, $H(t, t) = 0$, $H(t, t_0) \leq q(t_0)$ for some

$$q \in \mathscr{L} \quad \text{and} \quad \lim_{t \to \infty} [\sup_{t_0 \geq T_1} H(t . t_0)] = 0 \qquad \text{for} \quad T_1 > 0.$$

Clearly,

$$u(t, t_0, u_0) \leq M(t_0, \tau) u_0 + q(t_0), \qquad t \geq t_0$$

hence Theorem 3.1 implies that

$$\| y_t(t_0; \phi_0, \psi_0 \|_0 \leq K(t_0, \tau) M(t_0, \tau) (\| \phi_0 \|_0 + \| \psi_0 \|_0) + q(t_0), \qquad t \geq t_0$$

where $y_t(t_0; \phi_0, \psi_0)$, $z_t(t_0; \phi_0, \psi_0)$ is any solution of the perturbed system (2). Hence, using arguments similar to that of Theorem 3.1,

$$\| y_t(t_0; \phi_0, \psi_0) \|_0 \leq r(t, t_0, u_0) \leq M(t_0, \tau) u_0 e^{-\alpha(t-t_0)} + H(t, t_0)$$

for $t \geq t_0$, where $r(t, t_0, u_0)$ is the maximal solution of (4) such that $u_0 = K(t_0, \tau)(\|\phi_0\|_0 + \|\psi_0\|_0)$. Hence,

$$\|y_t(t_0; \phi_0, \psi_0)\|_0 \leq M(t_0, \tau) K(t_0, \tau)(\|\phi_0\|_0 + \|\psi_0\|_0) e^{-\alpha(t-t_0)} + H(t, t_0) \qquad t \geq t_0 .$$

Moreover,

$$\|y_t(t_0; 0, 0)\|_0 \leq H(t, t_0) \leq q(t_0)$$

for some $q \in \mathscr{L}$ and $t \geq t_0$, so that the set $\phi = 0$, $\psi = 0$ is ϕ-ASI with respect to the perturbed system (2) and it is partially equi-exponential asymptotically stable.

THEOREM 3.3. *Assume that the hypothesis of Theorem 2.8 of* [1] *hold. In addition, let the hypothesis*

(ii) *of Theorem 3.1 also hold. Then the ASI set* $u = 0$ *of*

$$\frac{du}{dt} = -p'(t) u + \eta(t) + K(t, \tau) \omega(t, u) \tag{5}$$

is generalized equi-exponential asymptotically stable implies that the ϕ*-ASI set* $\phi = 0$, $\psi = 0$ *of the perturbed system* (2) *is generalized partially equi-exponential asymptotically stable.*

Proof. In view of Theorem 2.8 of [1], arguments similar to that of Theorem 3.2 yield the required result. We omit details.

COROLLARY 3.4. *The* ϕ*-ASI set* $\phi = 0$, $\psi = 0$ *of* (2) *is generalized equi-exponential asymptotically stable with respect to the y-component, if* $w(t, u) = \lambda(t) u$ *where*

$$p(t_0) - p(t) + \int_{t_0}^{t} (k(s) \lambda(s) + \eta(s)) \, ds \to -\infty \qquad \text{as} \quad t \to \infty .$$

Proof. The general solution of (5) is

$$u(t, t_0, u_0) = u \exp\left(p(t_0) - p(t) + \int_{t_0}^{t} (k(s) \lambda(s) + \eta(s)) \, ds\right), \qquad t \geq t_0 .$$

By Theorem 3.3 the result follows.

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