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Analisi funzionale. — *On the existence of an unbounded connected set of solutions for nonlinear equations in Banach spaces.* Nota (*) di MASSIMO FURI e MARIA PATRIZIA PERA (**), presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia $f: E \to F$ un'applicazione continua fra spazi di Banach. Nel presente lavoro si danno delle condizioni su f affinchè l'equazione $f(x) = 0$ ammetta una componente connessa non limitata di soluzioni. A tale scopo si introducono le nozioni di applicazione o–regolare e di applicazione o–regolarizzabile. I risultati astratti sono infine applicati ad alcuni problemi ai limiti per equazioni differenziali a derivate parziali ed ordinarie.

0. INTRODUCTION

Let $f: E \to F$ be a continuous map from a Banach space E into a Banach space F. The purpose of this paper is to investigate, under suitable assumptions on f, the set $f^{-1}(0)$ of solutions of the equations $f(x) = 0$. In particular we give conditions on f which ensure that the set $f^{-1}(0)$ contains an unbounded connected component. To this aim we introduce the notion of o–regularizable (zero-regularizable) map, which turns out to be a suitable nonlinear extension of the notion of bounded linear surjective operator. All the proofs given in this paper avoid degree arguments and are mainly based on the very elementary theory of o-epi maps introduced in [2] and, consequently, on the most important "existence tool" for nonlinear operator equations in Banach spaces: Schauder's fixed point theorem.

As far as we know only few papers contain results in this (or similar) direction and all of them are based on degree theory (see e.g. [1], [3], [5]).

1. DEFINITIONS AND PRELIMINARY RESULTS

Let E , F be real Banach spaces and let X be a subset of E. A continuous map $g: X \to F$ is said to be *compact* if it maps bounded sets of X into relatively compact subsets of F and is said to be *proper* if $g^{-1}(K)$ is compact for every compact subset $K \subset F$. It is easy to see that a proper map sends closed sets of X into closed sets of F. Moreover, if $g: X \to F$ is proper and $h: X \to F$

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is such that $\overline{h(X)}$ is compact, then $g+h$ is proper. In particular, if $g: X \to F$ is proper on bounded closed sets of X and $h: X \to F$ is compact, then $g+h$ is proper on bounded closed sets of X. Furthermore it is known that a bounded linear operator $L: E \to F$ is proper on bounded closed sets of E if and only if $\dim \operatorname{Ker} L < +\infty$ and $Im\, L = \overline{Im\, L}$.

Let us consider now an open subset $U \subset E$ and a continuous map $f: \overline{U} \to F$. We say that f is *p-admissible* ($p \in F$) if $f^{-1}(p)$ is a bounded subset of U. Recall (see [2]) that a p-admissible map is called p-epi if the equation $f(x) - p = h(x)$ has a solution in U for any compact map $h: \overline{U} \to F$ with bounded support and such that $h(x) = 0$ for all $x \in \partial U$. It follows immediately that, if f is p-epi, then the equation $f(x) = p$ is solvable in U.

We say that a continuous map $H: \overline{U} \times [0, 1] \to F$ is a *0-homotopy* if

a) the set $S = \{x \in \overline{U} : H(x, \lambda) = 0 \text{ for some } \lambda \in [0, 1]\}$ is a bounded subset of U,

b) the map $(x, \lambda) \mapsto H(x, \lambda) - H(x, 0)$ from $\overline{U} \times [0, 1]$ into F is compact.

Moreover, $f, g: \overline{U} \to F$ are said to be 0-homotopic if there exists a 0-homotopy $H: \overline{U} \times [0, 1] \to F$ joining them (i.e. $H(\cdot, 0) = f$, $H(\cdot, 1) = g$).

The following theorems concerning 0-epi maps will be useful in the sequel (see [2]).

THEOREM 1.1. (*homotopy property*). *Let $f, g: \overline{U} \to F$ be 0-homotopic. Then f and g are either both 0-epi or both not 0-epi.*

THEOREM 1.2. (*normalization property*). *Let $f: \overline{U} \to F$ be continuous, injective and proper. Assume that $f(U)$ is open. Then f is p-epi if and only if $p \in f(U)$.*

THEOREM 1.3. (*localization property*). *Let $f: \overline{U} \to F$ be 0-epi and let V be an open subset of U containing $f^{-1}(0)$. Then $f \mid \overline{V}$ is 0-epi.*

2. 0-REGULAR AND 0-REGULARIZABLE MAPS

Let U be an open subset of E and let $f: \overline{U} \to F$ be a continuous map. We say that f is 0-*regular if it is* 0-*epi and proper on bounded closed subsets of* $\overline{U}$.

We introduce the following

DEFINITION 2.1 *Let $f: \overline{U} \to F$ be continuous. Assume that there exists a continuous map $\varphi: \overline{U} \to G$, G Banach space, such that*

i) *for any bounded subset $A \subset \overline{U}$, $\varphi(A)$ is bounded in G*

ii) *the map $g = (f, \varphi): \overline{U} \to F \times G$ defined by $g(x) = (f(x), \varphi(x))$ is 0-regular; then we say that f is 0-regularizable (by φ) or, equivalently, that the map φ 0-regularizes f.*

Clearly, f is 0-regular if and only if it is 0-regularizable by the trivial map $\varphi: \overline{U} \to \{0\}$.

If f is o-regularizable and not o-epi, then f is called *nontrivially* o-regularizable. As we remarked above, if f is nontrivially o-regularizable, then necessarly, we have $G \neq \{0\}$.

Finally a o-regularizable map $f: \overline{U} \to F$ is o-regular if and only if f is o-admissible and proper on bounded closed subsets of $\overline{U}$. Indeed, if f is o-regular, then by definition is it o-admissible and proper. Conversely, since f is o-regularizable, there exists $\varphi: \overline{U} \to G$ such that $(f, \varphi): \overline{U} \to F \times G$ is o-regular. Let $h: \overline{U} \to F$ be a compact map with bounded support, $h(x) = 0$ for all $x \in \partial U$. Clearly the equation $(f(x), \varphi(x)) = (h(x), 0)$ is solvable in U, and this implies, in particular, that f is o-epi.

The following result is related with the study of the solution-set of some types of differential equations, as we will show in the examples of section 4.

THEOREM 2.1. *Let* $f: \overline{U} \to F$ *be nontrivially* o-*regularizable by* $\varphi: \overline{U} \to G$. *Then there exists a connected component* Σ *of* $f^{-1}(0)$ *intersecting* $\varphi^{-1}(0)$ *and verifying at least one of the following two conditions*:

a) Σ *is unbounded*;

b) $\Sigma \cap \partial U \neq \varnothing$.

The following Lemma will be used in the proof of Theorem 2.1.

LEMMA 2.1. (*see* [6]) *Let* X *be a compact metric space and let* A *and* B *be closed disjoint subsets of* X. *Then either there exists a component of* X *which connects* A *and* B, *or there exist two closed sets* X_A *and* X_B *containing* A *and* B *respectively and such that* $X_A \cap X_B = \varnothing$, $X_A \cup X_B = X$.

Proof of Theorem 2.1. Denote by S the solution-set of the equation $f(x) = 0$ and observe that $S_0 = S \cap \varphi^{-1}(0) = (f, \varphi)^{-1}(0, 0)$ is a compact subset of U. We shall consider two cases: (A) the set U is bounded; (B) the set U is unbounded.

(A) *The set* U *is bounded.* Assume preliminarly that f is proper. Let $S_1 = S \cap \partial U$; we will show that $S_1 \neq \varnothing$. Since $G \neq \{0\}$, there exists $z_0 \in G$, $z_0 \neq 0$. For all $t \in \mathbf{R}$ define $g_t(x) = (f(x), \varphi(x) - tz_0)$. From the boundedness of $\varphi(\overline{U})$ in G we get that there exists $\bar{t} \in \mathbf{R}$ such that $\bar{t}z_0 \notin \varphi(\overline{U})$; hence $g_{\bar{t}}(x) \neq (0, 0)$ for all $x \in \overline{U}$. Consequently, there exists $t_0 \in]0, \bar{t}[$ such that the equation $g_{t_0}(x) = (0, 0)$ is solvable on ∂U. In fact, if this is not the case, we have $g_t(x) \neq (0, 0)$ for all $x \in \partial U$ and $t \in]0, \bar{t}[$; thus Th. 1.1 implies that $g_{\bar{t}}$ is o-epi, contradicting $g_{\bar{t}}(x) \neq (0, 0)$ for all $x \in \overline{U}$. Therefore, in particular $f(x) = 0$ has a solution on ∂U, i.e. $S_1 \neq \varnothing$.

We are now able to apply Lemma 2.1 to S_0 and S_1 which are closed disjoint subsets of the compact metric space $S = f^{-1}(0)$ (recall that f is assumed to be proper). If it does not exist a component of S connecting S_0 with S_1, then by Lemma 2.1 one can find F_0 and F_1, closed subsets of S (and hence closed subsets of $\overline{U}$), such that $F_0 \supset S_0$, $F_1 \supset S_1$, $F_0 \cap F_1 = \varnothing$, $F_0 \cup F_1 = S$.

Thus, there exists an open neighborhood V_0 of F_0 in U such that $\overline{V}_0 \cap F_1 = \varnothing$.

Therefore, as above, the equation $f(x) = 0$ has a solution $\bar{x} \in \partial V_0$ (observe that $V_0 \supset S_0$ and so, by the localization property of 0-epi maps, (f, φ) is 0-regular on V_0). Since $\bar{x} \in S$, we have $\bar{x} \in \partial V_0 \cap S = (\partial V_0 \cap F_0) \cup (\partial V_0 \cap F_1) = \varnothing$.

A contradiction. Thus, we can find a component Σ of S which connects S_0 and ∂U.

Let us remove now the assumption that f is proper on $\overline{U}$ In order to do this, let G_1 be a 1-codimensional subspace of G; so $G = G_1 \oplus \mathbf{R}x_0$, $x_0 \notin G_1$ and $\varphi = (\varphi_1, \varphi_2) : \overline{U} \to G_1 \times \mathbf{R}x_0$. Terefore $\hat{f} = (f, \varphi_1) : \overline{U} \to F \times G_1$ is 0-regularizable. Moreover, since $(\hat{f}, 0) = (\hat{f}, \varphi_2) - (0, \varphi_2)$ with $(\hat{f}, \varphi_2)$ proper and $(0, \varphi_2)$ compact. then $\hat{f}$ is a proper map. Consequently, by the previous argument, there exists a connected component of $\hat{f}^{-1}(0, 0)$ intersecting both S_0 and ∂U. Thus, the same is true for $f^{-1}(0) \supset \hat{f}^{-1}(0, 0)$.

(B) *The set* U *is unbounded*. Let $n_0 \in \mathbf{N}$ be such that the open ball centered at the origin with radius n_0 contains S_0. By (A) and the localization property of 0-epi maps (see Theorem 3.1), it follows that for any $n > n_0$ there exists $x_n \in S_0$ and a component $\Sigma_n \subset S$ such that Σ_n connects x_n with $\partial(B(0, n) \cap U) \subset \partial B(0, n) \cup \partial U$. By the compactness of S_0, the sequence $\{x_n\}$ has a cluster point $\bar{x} \in S_0$. Assume now that the connected component Σ of S containing $\bar{x}$ does not intersect ∂U. We want to prove that Σ is unbounded. Assume the contrary. Let D be an open bounded subset of U such that $D \supset \Sigma \cup S_0$. By (A) we get that $S \cap \partial D \neq \varnothing$; thus $\{\bar{x}\}$ and $S \cap \partial D$ are two closed disjoint subsets of $S \cap \bar{D}$ which, by assumption, cannot be connected by a component of S. Therefore, by Lemma 2.1, there exists an open neighborhood V_0 of $\bar{x}$ in D such that $\partial V_0 \cap S = \varnothing$. On the other hand there exists $\bar{n} \geq n_0$ such that $x_{\bar{n}} \in V_0$ and $B(0, \bar{n}) \supset D$. Consequently the connected set $\Sigma_{\bar{n}}$ intersects both V_0 and $E \setminus V_0$. Hence we get $\Sigma_{\bar{n}} \cap \partial V_0 \neq \varnothing$, contradicting the fact $S \cap \partial V_0 = \varnothing$.

Q.E.D.

Notice that the assertion of Theorem 2.1 is true even under the apparently weaker assumption that there exists a Banach space $G \neq \{0\}$ such that $(f, \varphi) : \overline{U} \to F \times G$ is 0-regular. On the other hand, taking into account what we observed at the beginning of this section we get the following.

COROLLARY 2.1. *A map $f : \overline{U} \to F$ is nontrivially 0-regularizable if and only if f is 0-regularizable by a map $\varphi : \overline{U} \to G$ with $G \neq \{0\}$.*

Theorems 2.2 and 2.3 below are easy consequences of analogous results proved in [2] for 0-epi maps.

THEOREM 2.2. (*Perturbation theorem*). *Let $f : \overline{U} \to F$ be 0-regularizable and assume that* U *is bounded. Let $h : \overline{U} \times [0, 1] \to F$ be compact and such that $h(x, 0) = 0$ for any $x \in \overline{U}$. Then there exists $\varepsilon > 0$ such that $f(\cdot) - h(\cdot . \lambda)$ is 0-regularizable for every $|\lambda| < \varepsilon$.*

THEOREM 2.3. (*Continuation principle*). *Let* $f : \overline{U} \to F$ *be* 0*-regularizable and let* $h : \overline{U} \times [0\,,1] \to F$ *be compact and such that* $h(x\,,0) = 0$ *for any* $x \in \overline{U}$. *Assume that there exists a map* $\varphi : \overline{U} \to G$ *which* 0*-regularizes* f *and such that the set* $S_\varphi = \{x \in \varphi^{-1}(0) : f(x) = h(x\,,\lambda)$ *for some* $\lambda \in [0\,,1]\}$ *is bounded and does not intersect* ∂U; *then* $f - h(\cdot\,,1)$ *is* 0*-regularizable* (*by* φ).

It is well known that a multivalued map $g : X \to Y$, X and Y metric spaces, is *upper semicontinuous* if and only if

i) the graph of g is closed in $X \times Y$

ii) g sends compact sets into relatively compact sets.

Using this fact and Theorem 1.1 of [2] we get the following.

THEOREM 2.4. *Let* $f : \overline{U} \to F$ *be* 0*-regular. Then there exists a neighborhood* V *of the origin in* F *such that* $f(U) \supset V$. *Moreover the multivalued map* $f^{-1} : V \to U$, *defined by* $f^{-1}(y) = \{x \in U : f(x) = y\}$, *is upper semicontinuous with compact values. In particular, if* f *is injective, then* f^{-1} *is continuous.*

3. NONLINEAR PERTURBATION OF LINEAR OPERATORS

Let $L : E \to F$ be a bounded linear operator. In the context of linear operators we have the following characterizations of 0-regular and 0-regularizable maps:

THEOREM 3.1. (*see* [2]). *A bounded linear operator* $L : E \to F$ *is* 0*-regular if and only if it is an isomorphism.*

THEOREM 3.2. *A bounded linear operator* $L : E \to F$ *is* 0*-regularizable if and only if it is onto.*

Proof (Only if). From the hypothesis it follows that there exists $\varphi : E \to G$ such that the map $g = (L\,,\varphi) : E \to F \times G$ is 0-regular. Suppose L not onto and take $p \in F$ such that $L^{-1}(p) = \varnothing$. We have $L^{-1}(\lambda p) = \varnothing$ for every $\lambda \in]0\,,1]$ and, consequently, $g^{-1}((\lambda p\,,0)) \subset L^{-1}(\lambda p) = \varnothing$ for all $\lambda \in]0\,,1]$. Moreover, being g 0-admissible, we get that $g^{-1}((0\,,0))$ is bounded. Thus the set $S = \{x \in E : g(x) = (\lambda p\,,0)$ for some $\lambda \in [0\,,1]\}$ is bounded. Hence, by the homotopy property for 0-epi maps (see Theorem 1.1), we get that $g(x) = (p\,,0)$ has a solution in E.

(If). By Michael's selection theorem [4] there exists a continuous map $s : F \to E$ such that $s(y) \in L^{-1}(y)$ for any $y \in F$ and $\| s(y) \|_E \leq M \| y \|_F$ for some $M > 0$. Let $\varphi : E \to E$ be defined by $\varphi(x) = x - s(Lx)$. Notice that $\mathrm{Im}\,\varphi \subset \mathrm{Ker}\,L$ since $L\varphi(x) = Lx - Ls(Lx) = 0$ for all $x \in E$. Moreover φ sends bounded sets of E into bounded sets of Ker L and the map $h : F \times \mathrm{Ker}\,L \to E$ defined by $h(y\,,z) = s(y) + z$ is the continuous inverse of $g = (L\,,\varphi) : E \to F \times \mathrm{Ker}\,L$. Thus g is a homeomorphism. Therefore, by Theorem 1.2, g is 0-regular.

Q.E.D.

The following two propositions exhibit some classes of problems which possess an unbounded and connected set of solutions.

PROPOSITION 3.1. *Let* $L : E \to F$ *be an isomorphism and let* $h : E \times \mathbf{R}^n \to F$ *be a compact map such that for all* $x \in E$, $\| h(x, 0) \| \leq a + b \| x \|^{\alpha}$, $a, b \geq 0$, $0 \leq \alpha < 1$. *Denote by* $S = \|(x, \lambda) \in E \times \mathbf{R}^n : Lx = h(x, \lambda)\}$. *Then there exists an unbounded connected component* $\Sigma \subset S$.

Proof. Let $\varphi : E \times \mathbf{R}^n \to \mathbf{R}^n$ be the map defined by $\varphi(x, \lambda) = \lambda$ and let $\hat{L} : E \times \mathbf{R}^n \to F$ be such that $\hat{L}(x, \lambda) = Lx$. By Theorem 3.1 $(\hat{L}, \varphi) : E \times \mathbf{R}^n \to F \times \mathbf{R}^n$ is 0-regular. Moreover it is easy to see that the set $\{(x, \lambda) \in E \times \mathbf{R}^n : (\hat{L}, \varphi)(x, \lambda) = \tau(h(x, \lambda), 0)$ for some $\tau \in [0, 1]\}$ is bounded. Therefore; by the continuation principle (Theorem 2.3), $\hat{L} - h$ is 0-regularizable and the assertion follows from Theorem 2.1.

Q.E.D.

PROPOSITION 3.2. *Let* $L : E \to F$ *be bounded linear and surjective with* $0 < \dim \operatorname{Ker} L < +\infty$. *Let* $h : E \to F$ *be compact and such that, for any* $x \in E$, $\| h(x) \| \leq a + b \| x \|^{\alpha}$, $a, b \geq 0$, $0 \leq \alpha < 1$. *Then* $(L - h)^{-1}(0)$ *contains a connected component which is unbounded.*

Proof. Since $\dim \operatorname{Ker} L < +\infty$, there exists a bounded linear projector P onto Ker L which 0-regularizes L. Now, observe that the set $\{x \in E : Px = 0$ and $Lx = \lambda h(x)$ for some $\lambda \in [0, 1]\}$ is bounded. Thus $L - h$ is 0-regularizable (by P) and the assertion follows from Theorem 2.1.

Q.E.D.

4. EXAMPLES

Example 4.1. Let Ω be a bounded domain in $\mathbf{R}^n$ whose boundary is an $(n - 1)$-dimensional smooth submanifold of $\mathbf{R}^n$. For every nonnegative integer k and for $\mu \in]0, 1[$, denote by $C_0^{k+\mu}(\overline{\Omega})$ the vector subspace of those functions $u \in C^k(\overline{\Omega})$, vanishing on $\partial\Omega$, and such that all the partial derivatives of order k are μ-Hölder continous on $\overline{\Omega}$. $C_0^{k+\mu}(\overline{\Omega})$ is a Banach space endowed with the norm

$$\| u \|_{k+\mu} = \| u \|_k + \sum_{|\alpha|=k} \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{| D^{\alpha} u(x) - D^{\alpha} u(y) |}{\| x - y \|^{\mu}}$$

($\| \cdot \|_k$ is the usual norm in $C^k(\overline{\Omega})$).
Consider the nonlinear boundary value problem:

$$(4.1.1) \qquad \begin{cases} \Delta u = \lambda f(x, u) & \text{in } \Omega, \lambda \in \mathbf{R} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is continuously differentiable.

By a solution of (4.1.1) we mean a pair $(u, \lambda) \in C_0^{2+\mu}(\overline{\Omega}) \times \mathbf{R}$ sotifying (4.1.1). To reformulate our problem in an abstract setting, observe first that the map $\hat{h} : C^1(\overline{\Omega}) \times \mathbf{R} \to C^1(\overline{\Omega})$ defined by $\hat{h}(u, \lambda) = \lambda f(\cdot, u(\cdot))$ is continuous.

On the other hand, $C^{2+\mu}(\overline{\Omega})$ is compactly imbedded in $C^1(\overline{\Omega})$; thus one can regard $\hat{h}$ as a compact map $h : C_0^{2+\mu}(\overline{\Omega}) \times \mathbf{R} \to C^{\mu}(\overline{\Omega})$. Moreover it is well-known that $L = \Delta : C_0^{2+\mu}(\overline{\Omega}) \to C^{\mu}(\overline{\Omega})$ is an isomorphism. Hence, problem (4.1.1) can be rewritten as follows

$$Lu = h(u, \lambda) \tag{4.1.2}$$

Thus, by Proposition 3.1, equation (4.1.2) has an unbounded connected component of solutions in $C_0^{2+\mu}(\overline{\Omega}) \times \mathbf{R}$.

Example 4.2. Consider in $C^2[0, 1]$ the problem

$$\begin{cases} \ddot{x} = f(t, x) \\ \displaystyle\int_0^1 x(t)\, dt = 0 \end{cases} \tag{4.2.1}$$

where $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is a continuous map such that $|f(t, s)| \leq M$ for some $M > 0$.

Denote by E the set of all functions $x \in C^2[0, 1]$ such that $\int_0^1 x(t)\, dt = 0$ and let $F = C^0[0, 1]$.

Clearly the linear operator $L : E \to F$ defined by $(Lx)(t) = \ddot{x}(t)$ is onto with 1-dimensional kernel. Moreover, since E is compactly imbedded in F, the Nemystkij operator $h : E \to F$ given by $h(x)(t) = f(t, x(t))$ is compact and has bounded image because of the assumption $|f(t, s)| \leq M$. Then it follows immediately from Proposition 3.2, that the equation $Lx = h(x)$ admits an unbounded connected component of solutions in E.

Example 4.3. Consider in $C^1[0, 1] \times \mathbf{R}$ the problem:

$$\begin{cases} \dot{x} = \lambda f(t, x) \\ x^3(0) = x(1) \end{cases} \tag{4.3.1}$$

where $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is continuous. By a solution we mean a pair $(x, \lambda) \in C^1[0, 1] \times \mathbf{R}$ satisfying (4.3.1). We want to show that the solution-set of our problem contains an unbounded connected component. In order to see this, let $G : C^1[0, 1] \times \mathbf{R} \to C^0[0, 1] \times \mathbf{R}$ be defined by $G(x, \lambda) = (\dot{x} - \lambda f(\cdot, x(\cdot)), x^3(0) - x(1))$ and let $\varphi : C^1[0, 1] \times \mathbf{R} \to \mathbf{R}$ be such that $\varphi(x, \lambda) = \lambda$. It is enough to prove (see Theorem 2.1) that φ 0-regularizes G. Consider $G_1 : C^1[0, 1] \times \mathbf{R} \to C^0[0, 1] \times \mathbf{R}$, $G_1(x, \lambda) = (\dot{x} - \lambda f(\cdot, x(\cdot)), x(0))$ and let $H : C^1[0, 1] \times \mathbf{R} \times [0, 1] \to C^0[0, 1] \times \mathbf{R} \times \mathbf{R}$ be the homotopy $H((x, \lambda), \tau) = (\dot{x} - \lambda f(\cdot, x(\cdot)), \lambda, \tau x(0) + (1 - \tau)(x^3(0) - x(1)))$. Clearly, the set

$\{(x, \lambda) \in C^1[0, 1] \times \mathbf{R} : H((x, \lambda), \tau) = (0, 0, 0)$ for some $\tau \in [0, 1]\} = \{(x, \lambda)$: $\dot{x} - \lambda f(\cdot, x(\cdot)) = 0, \lambda = 0, \tau x(0) + (1 - \tau)(x^3(0) - x(1)) = 0$ for some $\tau \in [0, 1]\} = \{(x, \lambda) : x(t) = c = \text{const.}$ for all $t \in [0, 1], \tau c + (1 - \tau)(c^3 - c) = 0$ for some $\tau \in [0, 1]\}$ is bounded ad $H(\cdot, 0) = (G, \varphi), H(\cdot, 1) = (G_1, \varphi)$. Thus (G, φ) is 0-regular if and only if (G_1, φ) is 0-regular. Now, observe that (G_1, φ) can be regarded as the sum of the linear isomorphism $L : C^1[0, 1] \times \mathbf{R} \to C^0[0, 1] \times \mathbf{R} \times \mathbf{R}$ defined by $L(x, \lambda) = (\dot{x}, x(0), \lambda)$ and of the compact map $h : C^1[0, 1] \times \mathbf{R} \to C^0[0, 1] \times \mathbf{R} \times \mathbf{R}$ defined by $h(x, \lambda) = -(\lambda f(\cdot, x(\cdot)), 0, 0)$. Therefore, since the set $\{(x, \lambda) : L(x, \lambda) = \tau(\lambda f(\cdot, x(\cdot)), 0, 0)$ for some $\tau \in [0, 1]\}$ is the singleton $\{0\}$, it follows from the Homotopy Property that $(G_1, \varphi) = L + h(\cdot, 1)$ is 0-regular.

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